

## DOCTOR OF PHILOSOPHY

### Generalisation of Clairaut's theorem to Minkowski spaces

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*Award date:*  
2013

*Awarding institution:*  
Coventry University

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# **Generalisation of Clairaut's Theorem to Minkowski Spaces**

By  
**Anis Saad**

**December 2013**



The work contained within this document has been submitted  
by the student in partial fulfilment of the requirement of their course and award

# **Generalisation of Clairaut's Theorem to Minkowski Spaces**

**By**

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**PHD**

**December 2013**

**Coventry University**

**A thesis submitted in partial fulfilment of the University's  
requirements for the Degree of Doctor of Philosophy in  
Mathematics.**



## DECLARATION

I confirm that this is my own work and the use of material from other sources has been properly and fully acknowledged.

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December 2013



## ABSTRACT

The geometry of surfaces of rotation in three dimensional Euclidean space has been studied widely. The rotational surfaces in three dimensional Euclidean space are generated by rotating an arbitrary curve about an arbitrary axis.

Moreover, the geodesics on surfaces of rotation in three dimension Euclidean space have been considered and discovered. Clairaut's [1713-1765]theorem describes the geodesics on surfaces of rotation and provides a result which is very helpful in understanding all geodesics on these surfaces.

On the other hand, the Minkowski spaces have shorter history. In 1908 Minkowski [1864-1909] gave his talk on four dimensional real vector space, with a symmetric form of signature  $(+, +, +, -)$ . In this space there are different types of vectors/ axes (space-like-time-like and null) as well as different types of curves (space-like- time-like and null).

This thesis considers the different types of axes of rotations, then creates three different types of surfaces of rotation in three dimensional Minkowski space, and generates Clairaut's theorem to each type of these surfaces, it also explains the analogy between three dimensional Euclidean and Minkowskian spaces.

Moreover, this thesis produces different types of surfaces of rotation in four dimensional Minkowski spaces. It also generalises Clairaut's theorem for these surfaces of rotations in four dimensional Minkowski space. Then we see how Clairaut's theorem characterization carries over to three dimensional and four dimensional Minkowski spaces.

## ACKNOWLEDGEMENTS

In the name of Allah, the Most Gracious and the Most Merciful. All praises to God for the strength and blessing in completing this thesis.

This thesis would not have been possible without the guidance and support of many individuals. I am deeply grateful to my supervisor, **Dr: Robert Low** for his guidance, motivation and endless support and encouragement during my doctoral studies. In addition to discussions on topics related to my dissertation, his enlightening conversations on a wide range of topics have been a constant source of inspiration to continuously learn and pursue fundamental research in mathematics. I am truly thankful to him for providing me the freedom and encouragement to pursue any topic of interest across various disciplines and make available to me many opportunities to broaden my technical, intellectual and personal skills.

Also, I'm grateful to the Faculty of Engineering and Computing and Applied Mathematics Research Centre secretaries for assisting me in many different ways.

I am indebted to the Omar Almokhtar university in Albayda Libya, who has faith in me to pursue my study and provided the financial support.

Finally, I'm forever grateful to my family, my beloved wife Najla Mustafa and my children for their understanding, endless patience and encouragement; when it was most required.





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## 1. INTRODUCTION

The relationship between Euclidean and Minkowskian geometry has many intriguing aspects, one of which is the manner in which formal similarity can co-exist with significant geometric disparity. There has been considerable interest in the comparison of these two geometries, as can be seen in the lecture notes of López [19].

In particular rotational surfaces in three dimensional Euclidean space have been studied for a long time and many examples of such surfaces have been discovered. The geodesics for surfaces of revolution also have been studied. One particular theorem, that is very helpful to understand the geodesics on surfaces of revolutions, is Clairaut's theorem. This gives a well-known characterization of geodesics on surfaces of revolutions. See, for example, Pressley's differential geometry textbook [3].

On the other hand, Minkowski space which we are more interested on in this thesis has a more complicated geometric structure compared to Euclidean space. In Minkowski space we can distinguish the types of the lines, space-like, time-like and null. In Euclidean space all straight lines are equivalent, so all types of rotations are equivalent. In Minkowski space however there are three distinct types of axes of rotations corresponding to the three classes of lines ;space-like, time-like and null. More explicitly, there are three types of one parameter sub-groups of isometries of three dimensional Minkowski spaces, each of which leave a line (axis) pointwise fixed. By considering the rigid motion of ambient space that keeps the straight line fixed, we investigate the corresponding rotation group for each. This lets us generate the matrices of rotation corresponding to each axis of rotation. Thus, in three dimensional Minkowski space, we are taking a time-like parametrized plane curve generating a surface of Lorentzian signatures. And then for each matrix of rotation we have

the special case of surfaces of rotation. Furthermore, we will see how Clairaut's theorem characterization carries over to Minkowski space. We will also investigate the differences between the two situations of three dimensional Euclidean and Minkowskian spaces.

In the case of four dimensional Minkowski space we need to seek two parameter subgroups of special orthogonal matrices of four dimensions  $SO(3,1)$  which are analogues of rotation in three dimensional Euclidean space; these fix three different types of axes of rotations. We again take a time-like parametrized plane curve, and then for each group of matrices of rotations we have special case of a Lorentzian surface of rotation. Moreover, see how Clairaut's theorem characterization carries over to four dimensional Minkowski space.

In chapter two we review some basic definitions and concepts of classical differential geometry of three dimensional Euclidean space. This chapter summarises the relevant part of information that can be found in any differential geometry textbook. The geometry of curves and surfaces, first and second fundamental forms, curvatures and geodesics. In particular we mention Clairaut's theorem on surfaces of revolution; which we will concentrate on in this thesis.

Chapter three provides background and literature review material. This chapter will include an introduction to Minkowski spaces and special relativity. Moreover, the concepts on analogy of differential geometry of Minkowski space will be demonstrated in the cases of curves and surfaces in general in Minkowski space.

In chapter four, the matrices of rotations in three dimensional Minkowski space has been generated. This chapter begins by reviewing the rotations in Euclidean space, and the isometries of Minkowski space. Considering the Killing vector field in three dimensional Minkowski space, rotations, boosts and null rotations are discussed. That leads to three types of matrices of rotations analogue to rotations in three dimensional Euclidean space. Therefore, we obtain three different types of matrices of rotations corresponding to the axes of rotation.

---

The surfaces of rotation corresponding to the matrices of rotations are provided in chapter five. This also provides the differential geometry of these types of surfaces of rotations in three dimensional Minkowski space and allows to generate Clairaut's theorem to three dimensional Minkowski space. Furthermore, we consider an explicit example and results to demonstrate the difference between the geodesics in Euclidean and Minkowskian spaces.

Chapter six is a brief description of the groups of matrices of rotations and surfaces of rotations in four dimensional Minkowski space. We recall Killing vector fields in four dimensional Minkowski space, and then generate two parameter subgroups of special orthogonal matrices of four dimensions  $SO(3, 1)$ ; these are analogues of rotations in Euclidean space. Thus these two parameter subgroups will be used to parametrize special surfaces of rotations in four dimensional Minkowski space.

In chapter seven, we generate Clairaut's theorem characterization over the surfaces of rotations of four dimensional Minkowski space with results.

At the end of this thesis, we give an overall conclusion and a future work plan.





## 2. BACKGROUND MATERIAL I : CURVES AND SURFACES IN EUCLIDEAN SPACE

This chapter reviews some basic definitions and conventions that can be found in standard textbooks and monographs on differential geometry such as [1],[2], [3] and[5].

We will benefit from this introduction in describing the differential geometry of Minkowski space.

For more complete elementary information about multi-linear algebra see, for example [4] chapters 7 and 8.

### 2.1 Curves and Surfaces

#### 2.1.1 Curves

**Definition 2.1.** *A curve is an immersion  $\gamma : I \rightarrow V$  defined on an open, possibly unbounded interval  $I \subset \mathbb{E}$ , where  $\mathbb{E}$  is 1D Euclidean space, into a vector space  $V \subset \mathbb{E}^n$ .*

A smooth curve is a differentiable function at least three times. Moreover the curve  $\gamma$  is called *regular and arclength parametrized* if  $g(\gamma', \gamma') = 1$ . Where  $g(\gamma', \gamma')$  is the inner product in  $\mathbb{E}^n$ .

Further, the *arc length* of this curve given by  $L_{t_0}^t(\gamma) = \int_{t_0}^t \sqrt{g(\gamma', \gamma')} du$ .

Any smooth curve  $\gamma : I \rightarrow V$  can be parametrized by arc length, see [1],[2], [3] and [5]. Then the curve is called a *regular parametrized curve*.

In general, henceforth we consider a regular parametrized curve, and assume that  $\gamma'', \gamma'''$  are exists and continuous with all  $\gamma', \gamma''$  and  $\gamma'''$  being linearly independent.

Some interesting, local theory of space curves is given by the concepts of *Frenet Frames*

a detailed treatment of which can be found in any differential geometry textbook e.g [3]. The Frenet frame is an orthonormal basis for  $\mathbb{E}^3$  chosen at each point on the curve, adapted to the geometry of the curve as much as possible. If the curve is arc-length parametrized, the vectors are  $T = \gamma'$ ,  $N = \frac{T'}{\|T'\|}$  and  $B = T \times N$ . Further  $\kappa$  and  $\tau$  define the curvature and the torsion of the curve in space.

These are related by the Frenet-Serret equation:

$$\begin{pmatrix} T'(s) \\ N'(s) \\ B'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}. \quad (2.1)$$

The Frenet frames is the best known frame adapted to a curve , but there are alternatives in their own strengths. The Bishop Frame [10] is an alternative approach to defining a moving frame. There is extensive literature on this subject. Such as [1],[10] and [12].

Let  $\gamma : I \rightarrow \mathbb{E}^3$  be a curve. Further, let  $N_1, N_2$  be two vector fields such that  $N_2 = T \times N_1$ ,  $T$  is tangent vector field. With  $g(T, N_1) = g(T, N_2) = g(N_1, N_2) = 0$ . Then we know  $T, N_1$  and  $N_2$  are an orthonormal frame as we move along the curve.

The Frenet frame works well if  $\kappa \neq 0$ . But if we require the alternative condition  $g(N_1', N_2) = 0$ , then the unit normal vector field  $N_1$  is *parallel* along the curve  $\gamma$ . Which means  $N_1'$  is in the direction of  $T$ . In this case.  $T, N_1, N_2$  are called a *Bishop Frame*, and  $\kappa_1$  and  $\kappa_2$  are Bishop curvatures.

Then, the Bishop Formula in  $\mathbb{E}^3$  is given in matrix form by:.

$$\begin{pmatrix} T'(s) \\ N_1'(s) \\ N_2'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1(s) & \kappa_2(s) \\ -\kappa_1(s) & 0 & 0 \\ -\kappa_2(s) & 0 & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N_1(s) \\ N_2(s) \end{pmatrix}. \quad (2.2)$$

where  $T'(s), N_1'(s)$  and  $N_2'(s)$  denote the derivative with respect to  $s$ .

## 2.1.2 Surfaces

To define a surface, we need the concepts of continuity and homomorphism of mapping from  $\mathbb{E}^m$  to  $\mathbb{E}^n$ . In  $\mathbb{E}^3$  for example, one can define the *Surface* as a subset  $\mathcal{S}$  of  $\mathbb{E}^3$ , if there is a non-empty open set  $U \subseteq \mathbb{E}^2$ , and an immersion

$$\sigma : U \rightarrow \mathcal{S}, \quad (2.3)$$

which provides a parametrization of  $\mathcal{S}$ .

A parametrization  $\sigma : U \rightarrow \mathcal{S}$  is said to be a *smooth*, if each component has continuous partial derivatives of all orders.

Also a parametrization  $\sigma$  is called *regular* if the tangent vector fields  $\sigma_u = \partial\sigma/\partial u$  and  $\sigma_v = \partial\sigma/\partial v$  are linearly independent for every point  $(u, v) \in U$ . Equivalently, the vector product  $\sigma_u \times \sigma_v \neq 0$ .

**Definition 2.2.** *A smooth surface is a surface  $\mathcal{S}$  whose parametrization consists of regular surface patches.*

Further explanation and details can be found in [1],[2], [3] and [5].

We are going to be interested in curves on surfaces, so we recall curves on surfaces.

**Proposition 2.3.** *If a smooth regular curve  $\gamma : (a, b) \rightarrow \mathbb{E}^3$  is contained in a surface whose parametrization is  $\sigma : U \rightarrow \mathcal{S}$ , there exist a map  $s \mapsto (u(s), v(s))$  such that*

$$\gamma(s) = \sigma(u(s), v(s)), \quad (2.4)$$

where the functions  $u(s)$  and  $v(s)$  are smooth with  $u'(s), v'(s) \neq 0$ . Conversely, if  $u(s)$  and  $v(s)$  are smooth, then (2.4) defines a smooth regular curve on the surface  $\mathcal{S}$ . See ([3], P85).

**Definition 2.4.** *The first fundamental form of a surface in  $\mathbb{E}^3$  is the expression*

$$Edu^2 + 2Fdudv + Gdv^2, \quad (2.5)$$

such that  $E = g(\sigma_u, \sigma_u)$ ,  $F = g(\sigma_u, \sigma_v)$  and  $G = g(\sigma_v, \sigma_v)$ .

Also, on tangent space written in terms of the basis  $\sigma_u, \sigma_v$ , it can be represented in this basis by the symmetric matrix:

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}. \quad (2.6)$$

The first fundamental form describes the intrinsic geometry of a surface. Furthermore, we will later examine it in the context of studying curves on surfaces in both Euclidean and Minkowskian Spaces.

Finally, we end this section by defining a *second fundamental form* and the curvature of surfaces.

**Definition 2.5.** *If  $\sigma : U \rightarrow \mathcal{S}$  is a parametrization of a surface in  $\mathbb{E}^3$ , then the unit vector normal to the surface at any point is given by:*

$$n = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} \quad (2.7)$$

As mentioned before, the first fundamental form describes the intrinsic geometry of a surface. The second fundamental form describes the extrinsic geometry of a surface. The surfaces can be determined by their first and second fundamental forms, e.g. curvature of surfaces. ([3], P159). In this thesis we only need the expression of the second fundamental form.

**Definition 2.6.** *The second fundamental form of a surface is the expression:*

$$Ldu^2 + 2Mdudv + Ndv^2 \quad (2.8)$$

where,  $L = g(\sigma_{uu}, n)$ ,  $M = g(\sigma_{uv}, n)$ ,  $N = g(\sigma_{vv}, n)$ .

Where  $\sigma_{uu} = \frac{\partial^2 \sigma}{\partial u^2}$ ,  $\sigma_{uv} = \frac{\partial^2 \sigma}{\partial u \partial v}$  and  $\sigma_{vv} = \frac{\partial^2 \sigma}{\partial v^2}$ .

Again, we can represent the second fundamental form in matrix form as:

$$\mathbb{I}_\sigma = \begin{pmatrix} L & M \\ M & N \end{pmatrix}. \quad (2.9)$$

Following to [3] chapter 8, we state the Gaussian and mean curvatures below.

**Definition 2.7.** *The Gaussian curvature of a surface in  $\mathbb{E}^3$  is the function:*

$$K = \frac{LN - M^2}{EG - F^2}. \quad (2.10)$$

**Definition 2.8.** *The mean curvature of a surface in  $\mathbb{E}^3$  is the function:*

$$H = \frac{1}{2} \frac{LG - 2MF + NE}{EG - F^2}. \quad (2.11)$$

## 2.2 Surfaces of Revolutions in $\mathbb{E}^3$

In this thesis we will be interested in surfaces of revolution in Minkowski space, so we recall the Euclidean version.

**Definition 2.9.** *Let  $I$  be an open interval of a real line. Also let  $\gamma(u) = (\rho(u), 0, h(u))$  be a regular parametrized plane curve with  $x = \rho(u) > 0$ ,  $z = h(u)$ . Then the surface of revolution is created by rotating the curve  $\gamma$  around the  $z$  axis yielding a surface parametrized by :*

$$\sigma(u, v) = (\rho(u) \cos v, \rho(u) \sin v, h(u)), u \in I, 0 \leq v \leq 2\pi. \quad (2.12)$$

Without loss of generality, we here assume that  $\gamma$  is unit speed so that the functions  $\rho, h$  have the property of  $(\rho')^2 + (h')^2 = 1$ , where  $\rho'$  and  $h'$  denote the derivative with respect to the parameter  $u$  of the functions  $\rho(u)$  and  $h(u)$ .

Curves of constant  $u$  are called *parallels*, and curves of constant  $v$  are called *meridians*.

So, the unit vector pointing along a meridian is given by:

$$\sigma_u = (\rho'(u) \cos v, \rho'(u) \sin v, h'(u)), \quad (2.13)$$

also, the natural vector pointing along a parallel is given by:

$$\sigma_v = (-\rho(u) \sin v, \rho(u) \cos v, 0). \quad (2.14)$$

Now let us use the coordinates  $(u, v)$  on the surface of revolution  $\mathcal{S}$ .

The first fundamental form is:

$$du^2 + \rho(u)dv^2. \quad (2.15)$$

We can easily compute the Gaussian and mean curvatures from the equations (2.10) and (2.11) respectively. Then they are given by:

$$K = -\frac{\rho''(u)}{\rho'(u)} \text{ and } H = \frac{1}{2} \left( \rho'(u)h''(u) - \rho''(u)h'(u) + \frac{h'(u)}{\rho(u)} \right) = \frac{1}{2} \left( \frac{h'(u)}{\rho(u)} - \frac{\rho''(u)}{h'(u)} \right), \quad (2.16)$$

where  $\rho'(u), \rho''(u)$ ,  $h'(u)$  and  $h''(u)$  denote to the derivative with respect to  $u$ .

### 2.3 Geodesics

Geodesics on surfaces are curves which are the analogues of straight lines in the plane. Lines can be locally thought of either as shortest curves or more generally straightest curves.

**Definition 2.10.** A curve  $\gamma(s)$  on a surface  $\mathcal{S}$  is called a geodesic if  $\gamma''(s) = 0$  or  $\gamma''(s)$  is perpendicular to the tangent plane.

Equivalently, a curve  $\gamma(s)$  on a surface  $\mathcal{S}$  is geodesic if  $\gamma''(s)$  is normal to the surface.

More extensive literature properties and notes about the geodesics on a surfaces can be found in [1], [2], [3] and [5].

### 2.3.1 Geodesics on Surfaces of Revolution

Let  $\gamma : I \rightarrow S$  be a curve given by  $\gamma(s) = (x(u(s), v(s)), y(u(s), v(s)), z(u(s)))$  which is an arc-length parametrized geodesic on a surface of revolution given above (2.12). We need the differential equations satisfied by  $(u(s), v(s))$ . Denote the differentiation with respect to  $s$  by an overdot.

From the Lagrangian :

$$L = \dot{u}^2 + \rho^2 \dot{v}^2 \quad (2.17)$$

we obtain the Euler-Lagrange equations

$$\ddot{u} = \rho \rho' \dot{v}^2, \quad \frac{d}{ds} (\rho \dot{v}^2) = 0, \quad (2.18)$$

so that  $\rho \dot{v}^2$  is a constant of the motion.

There are a couple of interesting special cases of geodesics on a surface of revolution.

**Proposition 2.11.** [3] *On a surface of revolution, every meridian is a geodesic. And a parallel  $u = u_0$  is geodesic if and only if  $d\rho/du = 0$  when  $u = u_0$ .*

This proposition only deals with these special cases. To understand the rest of geodesics; we need the following theorem; *Clairaut's Theorem*, which is very helpful for studying geodesics on surfaces of revolution.

Most of this thesis will be devoted to finding analogues and generalization of this theorem in Minkowski spaces.



## 2.4 Clairaut's Theorem



Fig. 2.1: Alexis Claude de Clairaut [1713-1765] [31]

As much of the following material will be on how this theorem transfers to other situations we give a detailed exposition of the proof.

Let  $\mathcal{S}$  be a surface of revolution, obtained by rotating the curve  $x = \rho(u), y = 0, z = h(u)$  about the  $z$ -axis, where we assume that  $\rho > 0$ , and  $\rho'(u)^2 + h'(u)^2 = 1$ . Then  $\mathcal{S}$  is parameterized by:

$$\sigma(u, v) = (\rho(u) \cos v, \rho(u) \sin v, h(u)) \quad (2.19)$$

and has the first fundamental form

$$I = \begin{pmatrix} 1 & 0 \\ 0 & \rho(u)^2 \end{pmatrix}. \quad (2.20)$$

We also have:

$$\sigma_u = \frac{\partial}{\partial u} \sigma(u, v) = \begin{pmatrix} \rho'(u) \cos(v) \\ \rho'(u) \sin(v) \\ h'(u) \end{pmatrix} = \mathbf{n}_u, \quad (2.21)$$

where  $\mathbf{n}_u$  is the unit vector pointing along meridians of  $\mathcal{S}$ .

And:

$$\sigma_v = \frac{\partial}{\partial v} \sigma(u, v) = \begin{pmatrix} -\rho(u) \sin(v) \\ \rho(u) \cos(v) \\ 0 \end{pmatrix} = \rho \begin{pmatrix} -\sin(v) \\ \cos(v) \\ 0 \end{pmatrix} = \rho \mathbf{n}_v, \quad (2.22)$$

where  $\mathbf{n}_v$  is the unit vector pointing along parallels of  $\mathcal{S}$ . Since  $g(\mathbf{n}_u, \mathbf{n}_v) = 0$  the two form an orthonormal basis, the unit vector  $\mathbf{z}$  in the tangent plane to  $\mathcal{S}$  at  $\sigma(u, v)$  is of the form  $\mathbf{n}_u \cos \theta + \mathbf{n}_v \sin \theta$  where  $\theta$  is the angle between  $\mathbf{z}$  and  $\mathbf{n}_u$ .

Now let  $\gamma(s)$  be a geodesic on  $\mathcal{S}$ , given by  $u(s)$  and  $v(s)$ , so that

$$\gamma(s) = \begin{pmatrix} \rho(u(s)) \cos(v(s)) \\ \rho(u(s)) \sin(v(s)) \\ h(u(s)) \end{pmatrix}. \quad (2.23)$$

From the first fundamental form, we have the Lagrangian (2.17):

$$L = \dot{u}^2 + \rho^2 \dot{v}^2, \quad (2.24)$$

so the Euler-Lagrange equations, The solutions of which are arc-length parametrised geodesics, are (see 2.18):

$$\begin{aligned} \ddot{u} &= \rho \rho' \dot{v}^2 \\ \frac{d}{ds} (\rho^2 \dot{v}) &= 0. \end{aligned} \quad (2.25)$$

But now, we also have:

$$\begin{aligned} \dot{\gamma} &= \dot{u} \sigma_u + \dot{v} \sigma_v \\ &= \dot{u} \mathbf{n}_u + \rho \dot{v} \mathbf{n}_v \\ &= \mathbf{n}_u \cos \theta + \mathbf{n}_v \sin \theta \end{aligned} \quad (2.26)$$

where  $\theta$  is the angle between  $\dot{\gamma}$  and a meridian.

Equating the components of  $\mathbf{n}_v$  in the latter two expressions, we see that  $\rho \dot{v} = \sin \theta$ , so that  $\rho^2 \dot{v} = \rho \sin \theta$ . Hence the second Euler-Lagrange equation is equivalent to  $\rho \sin \theta$  being a constant along  $\gamma$ .

Conversely, suppose that  $\gamma$  is a unit speed curve with  $\rho \sin \theta$  constant, and with  $\dot{u} \neq 0$ .

Then since:

$$\dot{u}^2 + \rho^2 \dot{v}^2 = L = 1 \quad (2.27)$$

differentiation of this with respect to  $s$ , gives:

$$\dot{u}\ddot{u} + \rho\rho'\dot{u}\dot{v}^2 + \rho^2\dot{v}\ddot{v} = 0. \quad (2.28)$$

And differentiation the second Euler-Lagrange equation with respect to  $s$  gives:

$$\frac{d}{ds}(\rho^2\dot{v}) = 0 = 2\rho\rho'\dot{u}\dot{v} + \rho^2\ddot{v} \quad (2.29)$$

multiplying (2.29) with  $v$  and subtracting from (2.28) yields:

$$\ddot{u} = \rho\rho'\dot{v}^2, \quad (2.30)$$

which is the first Euler-Lagrange equation.

□

This establishes Clairaut's theorem as follows, and we observe in passing that all meridians are geodesics.

**Theorem 2.12** ([3], 228). *Let  $\gamma$  be a geodesic on a surface of revolution  $\mathcal{S}$ , let  $\rho$  be the distance function of a point of  $\mathcal{S}$  from the axis of rotation, and let  $\theta$  be the angle between  $\gamma$  and the meridians of  $\mathcal{S}$ . Then  $\rho \sin \theta$  is constant along  $\gamma$ . Conversely, if  $\rho \sin \theta$  is constant along some curve  $\gamma$  in the surface, and if no part of  $\gamma$  is part of some parallel of  $\mathcal{S}$ , then  $\gamma$  is a geodesic.*

## 2.5 Noether's Theorem

This discussion is a special case of the theorem of Noether, which states that if a system has a continuous symmetry property, then there are corresponding quantities whose values

are conserved i.e. conserved quantity along trajectories which means in our case Clairaut's theorem. For more literature see for example [46, 47].

### Hamiltonian Formalism

We have used the Lagrangian Formalism to find geodesics. Instead of this, we could use the equivalent Hamiltonian formalism, defined as follows:

Given a Lagrangian  $L(q_i, \dot{q}_i)$ ,  $i = 1 \dots n$ , we define the momenta

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \quad (i = 1 \dots n) \quad (2.31)$$

and the Hamiltonian :

$$H(q^i, p_i) = \sum_{i=1}^n p_i \dot{q}_i(q^i, p_i) - L(q^i, \dot{q}^i(q^i, p_i)) \quad (2.32)$$

then the *Hamiltonian equations*:

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q^i} \quad (2.33)$$

are equivalent to the Euler-Lagrangian equation for  $L(q^i, \dot{q}^i)$ .

We also define the *Poisson bracket* of two functions  $f, g$  of the variables  $q, p$  by:

$$[f, g] = \sum_{\alpha=1}^n \left( \frac{\partial f}{\partial q_\alpha} \frac{\partial g}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha} \frac{\partial g}{\partial q_\alpha} \right). \quad (2.34)$$

It satisfies

$$[f, g] = -[g, f] \quad , \quad [f, [g, h]] + [g, [h, g]] + [h, [f, g]] = 0. \quad (2.35)$$

The second property is called the *Jacobi identity*. Also the coordinate functions  $(p_i, q_j)$  satisfy the canonical commutation relations.

$$[p_j, p_k] = 0 \quad , \quad [q_j, q_k] = 0 \quad , \quad [q_j, p_k] = \delta_{jk}, \quad (2.36)$$

this has the important consequence that, if  $f$  is unchanged by transformation generated by  $g$ , then  $g$  is also unchanged by the transformation generated by  $f$ .

It can be shown that

$$\dot{f} = [f, H] \quad (2.37)$$

and so that any constant of the motion  $f$  satisfies

$$[f, H] = 0. \quad (2.38)$$

There is a remarkable theorem, the Liouville Arnol'd theorem, which tells us that if we have in independent constants of motion whose Poisson brackets with each other all vanish, then the equations of motion can be solved exactly in terms of integrals and algebraic operations. In this case the system is said to be completely integrable [[44], P294 and P347].

We end this section by this example of Euclidean case.

**Example 2.13.** *In this case we have the Lagrangian as in (2.17):*

$$L = \dot{u}^2 + \rho^2(u)\dot{v}^2 \quad (2.39)$$

*which is constant along a geodesic, and Euler-Lagrange equation*

$$\frac{d}{ds}(2\rho^2(u)\dot{v}) = 0, \quad (2.40)$$

*so that  $\rho^2\dot{v}$  is constant, say  $\Omega$ . In this case the Lagrangian is equal to the Hamiltonian, and so the conserved quantity  $\rho^2(u)\dot{v}$  commutes with  $H$ . We therefore have two community conserved quantities, and so the system should be completely integrable.*

*In fact, we can easily find the geodesics, as follows*

$$\dot{v} = \Omega/\rho^2(u) \quad (2.41)$$

and so,

$$L = \dot{u}^2 + \Omega^2/\rho^2(u) \quad (2.42)$$

giving

$$\dot{u} = \sqrt{L - \Omega^2/\rho^2(u)} \quad (2.43)$$

rearranging of this

$$\frac{du}{\sqrt{L - \Omega^2/\rho^2(u)}} = dt \quad (2.44)$$

so that

$$t = \int \frac{du}{\sqrt{L - \Omega^2/\rho^2(u)}} + C_1. \quad (2.45)$$

This specifies  $u$  as a function of  $t$ .

We now return to  $\rho^2(u)\dot{v} = \Omega$ , and  $u$  is a function of  $t$  obtained above, then

$$\dot{v} = \Omega/\rho^2(u(t)) \quad (2.46)$$

then we conclude

$$v = \int \frac{\Omega}{\rho^2(u(t))} dt + C_2. \quad (2.47)$$

This gives both  $u$  and  $v$  explicitly in terms of integrals.



### 3. BACKGROUND MATERIAL II

#### MINKOWSKI SPACES AND GEOMETRY



Fig. 3.1: Hermann Minkowski [1864–1909] [43]

*” The views of space and time which I wish to develop have sprung from the soil of experimental physics. Therein lies their strength. They have a radical tendency. Henceforth space by itself, and time by itself, will fade away into mere shadows, and only a kind of union of the two will preserve an independent existence.*

*Hermann Minkowski, Conference talk in Cologne, 21 September 1908, published in Raum und Zeit, Physikalische Zeitschrift, 104, (1909). ” [6]*

#### 3.1 Minkowski Spaces

Special relativity is based on the principle that we can consider the space of three dimensions combined with time, which form a four-dimensional space  $\mathbb{M}^{3,1}$ , with a dot product  $g$  called the *Lorentz inner product*. This means in terms of the coordinates  $(x, y, z, t)$  the inner product has a negative sign.



In the field of geometry of physics, Minkowski Space replaces Euclidean space; while all dimensions of Euclidean space are space-like, in Minkowski space there is one additional time-like dimension [7].

**Definition 3.1.** The space  $\mathbb{M}^{3,1}$  is defined as a four dimensional vector space consisting of vectors  $u = \{(u_x, u_y, u_z, u_t) : u_x, u_y, u_z, u_t \in \mathbb{R}\}$ , with a dot product  $g$  given by :

$$g(u, v) = u_x v_x + u_y v_y + u_z v_z - u_t v_t \quad (3.1)$$

This space is called a Minkowski Space and  $g$  is called a Minkowskian or Lorentzian inner product.

In summary, the inner product  $g$  in  $\mathbb{M}^{3,1}$  is defined as a nondegenerate symmetric bilinear form. The indefinite inner product allows a classification of vectors with no analogy in  $\mathbb{E}^3$ .

**Definition 3.2.** The vector  $v \in \mathbb{M}^{3,1}$  is said to be :

1. Time-like if  $g(v, v) < 0$ ,
2. Space-like if  $g(v, v) > 0$ ,
3. light-like or null if  $g(v, v) = 0$ ; while  $v \neq 0$ .

Any two different vectors  $u$  and  $v$  in  $\mathbb{M}^{3,1}$  are said to be *orthogonal* if  $g(u, v) = 0$ . Also a vector  $v \in \mathbb{M}^{3,1}$  which satisfies  $g(v, v) = \pm 1$  is called a *unit vector*. Any basis for  $\mathbb{M}^{3,1}$  such as  $\{e_x, e_y, e_z, e_t\}$  is an *orthonormal basis* if it consists of mutually orthogonal unit vectors specifically,  $g(e_x, e_x) = g(e_y, e_y) = g(e_z, e_z) = 1$ , and  $g(e_t, e_t) = -1$ , while all other inner products are zeros.

Although the  $\mathbb{M}^{3,1}$  space is the most appropriate for studying physics, it is also interesting in considering the corresponding space of one dimension less. So we define three dimensional Minkowski space  $\mathbb{M}^{2,1}$  as follows:

**Definition 3.3.** The space  $\mathbb{M}^{2,1}$  is the three dimensional inner product space; with signature  $(+, +, -)$ , such that the basis  $\{e_x, e_y, e_t\}$  with  $g(e_x, e_x) = g(e_y, e_y) = -g(e_t, e_t) = 1$ , and all  $g(e_x, e_y) = g(e_x, e_t) = g(e_y, e_t) = 0$ . Then for any two vectors  $u = (u_x, u_y, u_t)$  and  $v = (v_x, v_y, v_t)$  in  $\mathbb{M}^{2,1}$  the inner product is given by:

$$g(u, v) = u_x v_x + u_y v_y - u_t v_t. \quad (3.2)$$

Again, any basis for  $\mathbb{M}^{2,1}$  such as  $\{e_x, e_y, e_t\}$  is an *orthonormal basis* because it consists of mutually orthogonal unit vectors specifically,  $g(e_x, e_x) = g(e_y, e_y) = 1$ , and  $g(e_t, e_t) = -1$ , while all other products are zero.

### 3.2 Minkowski Vector Product

In mathematics the cross product or vector product is a binary operation between two vectors in three-dimensional space. It results in a vector which is perpendicular to both of the vectors being multiplied and normal to the plane containing them. This section will discuss this operation in general, and then specialise to three dimensional Minkowski space  $\mathbb{M}^{2,1}$ .

**Definition 3.4.** If  $u, v \in \bigwedge^k V$  where  $V$  is a vector space with inner product  $g$ , with an oriented basis  $(e_1, \dots, e_k)$ , then the inner product of  $u, v$  is defined by :

$$g(u_1 \wedge \dots \wedge u_k, v_1 \wedge \dots \wedge v_k) = \det(M); M_{ij} = g(u_i, v_j), \quad (3.3)$$

where,  $u_i = e_{i_1} \wedge \dots \wedge e_{i_k}$ , and  $v_j = e_{j_1} \wedge \dots \wedge e_{j_k}$ , and linearly extended to arbitrary forms.

**Definition 3.5.** If  $u, v \in \bigwedge^k V$ , for an oriented inner product space  $V$ , with orthonormal basis  $\{e_1, e_2, \dots, e_k\}$ ; then the Hodge star  $\star u$  is defined by:

$$v \wedge \star u = g(u, v) e_1 \wedge e_2 \wedge \dots \wedge e_k, \quad (3.4)$$

where  $g(u, v)$  is the inner product defined above.

If  $k = 3$  we consider  $\mathbb{E}^3$ , with orthonormal basis  $\{e_x, e_y, e_z\}$ . On computing the Hodge star of  $\star e_x, \star e_y$  and  $\star e_z$  as an example, we have  $g(e_x, e_x) = g(e_y, e_y) = g(e_z, e_z) = 1$  and  $g(e_x, e_y) = g(e_x, e_z) = g(e_y, e_z) = 0$ . In the same way, on computing  $\star(e_x \wedge e_y)$  gives  $e_z$ . Also  $\star(e_x \wedge e_z), \star(e_y \wedge e_z)$  give  $e_y, e_x$  respectively.

In conclusion, for any two vectors  $v, w \in \mathbb{E}^3$ , one can easily check that  $\star(v \wedge w) = v \times w$ ; i.e. gives the usual cross product in three dimensional Euclidean space. This is expressed in the familiar determinant form by:

$$v \times w = \begin{vmatrix} e_x & e_y & e_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} \quad (3.5)$$

for any two vectors  $v, w$  in  $\mathbb{E}^3$ .

This gives a third vector  $v \times w$  which is perpendicular to both  $v, w$ .

We now see how this general idea can be used to determine the cross product in three dimensional Minkowski space.

Basically, we use the same idea in multilinear algebra and define the Hodge star product for any vectors in  $\mathbb{E}^3$ . See ([4]: P203-235). Now we will be using the oriented Hodge star in the sense of  $\mathbb{M}^{2,1}$ .

If we take any two vectors  $u, v \in \mathbb{M}^{2,1}$ , the orthonormal basis here is  $\{e_x, e_y, e_t\}$ . On computing the Hodge star of  $\star e_x, \star e_y$  and  $\star e_t$ , we get  $g(e_x, e_x) = g(e_y, e_y) = -g(e_t, e_t) = 1$  and  $g(e_x, e_y) = g(e_x, e_t) = g(e_y, e_t) = 0$ . In the same way, on computing  $\star(e_x \wedge e_y)$  gives  $e_t$ . Also  $\star(e_x \wedge e_t), \star(e_y \wedge e_t)$  give  $-e_y, -e_x$  respectively. We then define the  $\times$  by linear extension.

In conclusion, for any two vectors  $u, v \in \mathbb{M}^{2,1}$ , one can compute  $u \times v = \star(v \wedge w)$ ; gives the vector product in three dimensional Minkowski space. This can be expressed by the use

a less familiar determinant form :

$$u \times v = \begin{vmatrix} -e_x & -e_y & e_t \\ u^x & u^y & u^t \\ v^x & v^y & v^t \end{vmatrix}. \quad (3.6)$$

Henceforth, we will use this formula for any vector product in Minkowski space  $\mathbb{M}^{2,1}$ .

**Note:** The vector  $u \times v$  is perpendicular to both  $u, v$ , just as in  $\mathbb{E}^3$ .

### 3.3 Lorentz Transformation and Relativity

Lorentz transformation has a long history with huge literature. It can be found in any relativity textbook, such as [[14],[15]]. Here; we will review only the transformation definition, statements and basic properties in three dimensional Minkowski spaces.

The Lorentz group is the group of isometries of Minkowski space which preserve the origin. Then the Minkowski metric is given by:

$$\eta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (3.7)$$

The Lorentz transformation is supposed to keep this combination of time and space intervals invariant.

**Definition 3.6.** *The Lorentz transformation for any position (point) in Minkowski space is defined by:*

$$\tilde{S} = \Lambda S, \quad (3.8)$$

where:

$$\Lambda^T \eta \Lambda = \eta \quad (3.9)$$

where  $\Lambda^T$  is the transposed matrix to the matrix  $\Lambda$ , and  $\eta$  is the Minkowski metric.

We will consider some interesting special types of Lorentz transformation in  $\mathbb{M}^{2,1}$ . The first consists of the boost in direction of  $x$ ; of the three-dimensional Minkowski space:

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{t} \end{pmatrix} = \begin{pmatrix} \gamma & 0 & -v\gamma \\ 0 & 1 & 0 \\ -v\gamma & 0 & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ t \end{pmatrix}, \quad (3.10)$$

where,  $v \in (-1, 1)$  and  $\gamma = 1/\sqrt{1-v^2}$ .

or

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{t} \end{pmatrix} = \begin{pmatrix} \cosh(\theta) & 0 & \sinh(\theta) \\ 0 & 1 & 0 \\ \sinh(\theta) & 0 & \cosh(\theta) \end{pmatrix} \begin{pmatrix} x \\ y \\ t \end{pmatrix}, \quad (3.11)$$

in this case, each point  $(0, y, 0)$  is fixed for each  $\theta$ . In other words, The  $y$ -axis can be thought as an axis of rotation through a hyperbolic angle  $\theta$ .

Similarly we can easily define the other boost in direction of  $y$ .

We can also consider a regular rotation, for example:

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{t} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ t \end{pmatrix}, \quad (3.12)$$

in this case,  $(0, 0, t)$  is fixed. Here the axis of rotation is  $t$ .

Finally, we have the null rotation :

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{t} \end{pmatrix} = \begin{pmatrix} 1 & -\theta & \theta \\ \theta & 1 - \frac{\theta^2}{2} & \frac{\theta^2}{2} \\ \theta & -\frac{\theta^2}{2} & 1 + \frac{\theta^2}{2} \end{pmatrix} \begin{pmatrix} x \\ y \\ t \end{pmatrix}, \quad (3.13)$$

in this case, the line  $y = t, x = 0$  is fixed. Here the axis of rotation is  $y = t, x = 0$ .

These transformations will be of particular interest later. There we will give more details corresponding to rotation in  $\mathbb{E}^3$ .

### 3.4 Curves in Minkowski Spaces

A curve  $\gamma$  is defined by a map  $\gamma : I \longrightarrow \mathbb{M}^{2,1}$  and satisfies the usual conditions of curves in space. i.e.  $\gamma$  is a differentiable function, and all derivatives  $\gamma', \gamma''$  and  $\gamma'''$  are linearly independent vectors. Note that  $\|\gamma'\|$  may be zero even if  $\gamma' \neq 0$ .

In Minkowski space; the dot product is not positive definite. As before, there are three cases of vectors as well as three cases of curves. This affects how the arc length is defined.

**Definition 3.7.** A regular curve  $\gamma : I \longrightarrow \mathbb{M}^{2,1}$  is called

1. Space-like curve if  $g(\gamma', \gamma') > 0$  everywhere.
2. Time-like curve if  $g(\gamma', \gamma') < 0$  everywhere.
3. Null or (Light-like) if  $g(\gamma', \gamma') = 0$ , everywhere, and  $\gamma' \neq 0$ .

By choosing the parameter, a regular curve  $\gamma$  which is space-like or time-like can be parametrized by arc length in the sense that  $g(\gamma', \gamma') = \pm 1$  is valid everywhere. For the curve which is null everywhere this is not possible in general.

#### 3.4.1 Analogy of Frenet and Bishop Frames in $\mathbb{M}^{2,1}$

Lets denote  $\{T, N, B\}$  as a Frenet trihedron moving along the curve  $\gamma$  in  $\mathbb{M}^{2,1}$ , with curvature  $\kappa$  and torsion  $\tau$ , such that  $T$  is a tangent vector equal to  $\gamma'$ ,  $N$  is the normal vector equals  $\frac{T'}{|T'|}$ ; and  $B$  is the binormal vector equals  $T \times N$ . Note that  $\times$  is the vector product in 3D Minkowski space presented above (eq. 3.6).

$\{T, N, B\}$  are three vectors moving along a curve, and  $\mathbb{M}^{2,1}$  spanned by two space-like vectors and one time-like vector. Therefore, for any (orthonormal) frame trihedron  $\{T, N, B\}$  one of them is time-like, i.e. the time-like moving along  $g(T, T), g(N, N)$  and,  $g(B, B)$  such that there is one time-like vector and the other two space-likes.

Consequently, one can state the following cases of Frenet frames.

**Case (1)** if  $g(T, T) = -1$ ,  $g(N, N) = 1$ ,  $g(B, B) = 1$ .

Then;  $N' = \alpha T + \beta B$ . So  $g(N', T) = \alpha g(T, T) = -\alpha$ . Also  $g(N', T) = -g(N, T') = -\kappa$ . Therefore  $\alpha = \kappa$ . In addition,  $g(N', B) = \beta g(B, B) = \beta$ . Also  $g(N', B) = -g(N, B') = -(-\tau)$ . Therefore  $\beta = \tau$ . So the Frenet Formula is given by:

$$\begin{pmatrix} T'(s) \\ N'(s) \\ B'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ \kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}. \quad (3.14)$$

Similarly **Case (2)** if  $g(T, T) = 1$ ,  $g(N, N) = -1$ ,  $g(B, B) = 1$ , so the Frenet Formula for this case is given by:

$$\begin{pmatrix} T'(s) \\ N'(s) \\ B'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ \kappa(s) & 0 & -\tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}. \quad (3.15)$$

Similarly **Case (3)** if  $g(T, T) = 1$ ,  $g(N, N) = 1$ ,  $g(B, B) = -1$ , then the Frenet Formula for this case is given by:

$$\begin{pmatrix} T'(s) \\ N'(s) \\ B'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & -\tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}. \quad (3.16)$$

In summary the Frenet Formula in  $\mathbb{M}^{2,1}$  is subtly different to  $\mathbb{E}^3$  case given in (Chapter 2). Moreover, all cases in  $\mathbb{M}^{2,1}$  are different to each other. It is seen that the signature of  $\kappa$  and  $\tau$  are changing along the cases above [17].

Alternatively, the Bishop frames of a curve in  $\mathbb{M}^{2,1}$  also have an analogy in Euclidean space, and displays three cases of time-like vectors of the frame trihedron.

Suppose that  $\{T, N_1, N_2\}$  are a Bishop trihedron, with the curvatures  $\kappa_1, \kappa_2$ , defined in

$\mathbb{E}^3$  as in chapter 2.

So consider a curve  $\gamma : I \longrightarrow \mathbb{M}^{2,1}$ . As in the Frenet frames case, we will have a time-like vector among the  $\{T, N_1, N_2\}$ , with  $\{T', N'_1, N'_2\}$  defined as in the  $\mathbb{E}^3$  case in chapter 2. Then the following cases for the Bishop equation are given as follows.

**Case (1)** Let  $T$  is time-like. Then  $N_1, N_2$  are space-like, and the frame is given by :

$$\begin{pmatrix} T'(s) \\ N'_1(s) \\ N'_2(s) \end{pmatrix} = \begin{pmatrix} 0 & -\kappa_1(s) & -\kappa_2(s) \\ -\kappa_1(s) & 0 & 0 \\ -\kappa_2(s) & 0 & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N_1(s) \\ N_2(s) \end{pmatrix}. \quad (3.17)$$

Similarly, **Case (2)** if  $N_1$  is time-like, the Bishop Formulae for this case are:

$$\begin{pmatrix} T'(s) \\ N'_1(s) \\ N'_2(s) \end{pmatrix} = \begin{pmatrix} 0 & -\kappa_1(s) & \kappa_2(s) \\ -\kappa_1(s) & 0 & 0 \\ -\kappa_2(s) & 0 & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N_1(s) \\ N_2(s) \end{pmatrix}. \quad (3.18)$$

Finally, **Case (3)** if  $N_2$  is time-like , the Bishop Formulae for this case are:

$$\begin{pmatrix} T'(s) \\ N'_1(s) \\ N'_2(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1(s) & -\kappa_2(s) \\ -\kappa_1(s) & 0 & 0 \\ -\kappa_2(s) & 0 & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N_1(s) \\ N_2(s) \end{pmatrix}. \quad (3.19)$$

In summary, as with Frenet Formulae, the Bishop Formula is different from the Euclidean case, since the matrix entries here have different signs. Moreover, all cases in 3D Minkowski space are different to each other.

In spite of this, the relationship between Frenet and Bishop Frames has an analogue to  $\mathbb{M}^{2,1}$  too. This relationship is laid out in [11] and [12].

In three dimensional Euclidean space, for any unit length curve, the relationship between



Frenet frames and Bishop frames is given by:

$$\begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta(s) & \sin \theta(s) \\ 0 & -\sin \theta(s) & \cos \theta(s) \end{pmatrix} \begin{pmatrix} T(s) \\ N_1(s) \\ N_2(s) \end{pmatrix}. \quad (3.20)$$

where  $\sin \theta = \frac{\kappa_1}{\kappa}$  and  $\cos \theta = \frac{\kappa_2}{\kappa}$ . Also,  $\kappa(s) = \sqrt{\kappa_1^2 + \kappa_2^2}$ ,  $\theta(s) = \arctan \frac{\kappa_2}{\kappa_1}$  and  $\tau(s) = \theta'(s)$ .

Now, let  $\gamma(s) : I \longrightarrow \mathbb{M}^{2,1}$  be a curve in 3D Minkowski space, and let the Frenet frames and Bishop frames of this curve  $\{T, N, B\}$  and  $\{T, N_1, N_2\}$  respectively, and assume that either  $N_1$  or  $N_2$  is time-like, (say  $N_2$  is time-like).

$T$  is the common vector field between the two frames. Also  $T' = \kappa N$  and  $T' = \kappa_1 N_1 + \kappa_2 N_2$ . Using these equalities and the Bishop frame derivatives formula see ([12], P 5-7), we get

$$N = \frac{T'}{\kappa} = \frac{\kappa_1 N_1 + \kappa_2 N_2}{\kappa}. \quad (3.21)$$

Taking the cross product with  $T$  on both sides of equation (3.21), we find, see (3.6), then:

$$B = \frac{\kappa_1 N_2 + \kappa_2 N_1}{\kappa} \quad (3.22)$$

Assuming that  $N_1$  or  $N_2$  is a time-like vector. Then one can write  $\cosh \theta = \frac{\kappa_1}{\kappa}$  and  $\sinh \theta = \frac{\kappa_2}{\kappa}$  in (3.21) and (3.22).

Therefore, producing that:

$$\begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \theta(s) & \sinh \theta(s) \\ 0 & \sinh \theta(s) & \cosh \theta(s) \end{pmatrix} \begin{pmatrix} T(s) \\ N_1(s) \\ N_2(s) \end{pmatrix}. \quad (3.23)$$

This is the relationship between the Frenet and Bishop frames in  $\mathbb{M}^{2,1}$  for time-like curve where  $N_1$  or  $N_2$  is a time-like vector.

In conclusion, we can say that in the Bishop frame and the Frenet frames of the curve rotate about the vector  $T$  by the hyperbolic angle  $\theta$ , where  $\theta = \operatorname{arctanh} \frac{\kappa_2}{\kappa_1}$  is assumed.

For any unit speed space-like curve, since  $\kappa = T'$  we obtain then  $\kappa = \sqrt{\epsilon_1 \kappa_1 + \epsilon_2 \kappa_2}$ , such that  $\epsilon_1 = \pm 1$  depending on whether  $N_1$  is space-like or time-like, and  $\epsilon_2 = \pm 1$  depending on whether  $N_2$  is space-like or time-like. Moreover,  $\tau(s) = \epsilon_1 \theta'(s)$ ,  $\theta(s) = \operatorname{arctanh} \frac{\kappa_2}{\kappa_1}$ . [12]

As a result,  $\kappa_1$  and  $\kappa_2$  correspond to a cartesian coordinate system for the polar coordinates  $\kappa, \theta$  with  $\theta(s) = \int \tau(s) ds$ .

### 3.5 Surfaces in Minkowski Space

As one can define curves in Minkowski space, the theory of surfaces in Minkowski space can be developed. A regular surface is defined as in  $\mathbb{E}^3$  an immersion  $\sigma : U \rightarrow \mathcal{S}$ , i.e. a differentiable map such that  $\sigma_u \times \sigma_v \neq 0$ .

Again, we give only a brief review of the principal relevant ideas. More information on this topic can be found in [5, 17, 18].

As we have three types of vectors (Sec.3.2), there are also different types of tangent planes.

**Definition 3.8.** *The tangent plane of the surface is defined by:*

$$T_p \mathcal{S} = \operatorname{Span}\{\sigma_u(p), \sigma_v(p)\}, \quad (3.24)$$

where  $p$  is a point in  $\mathcal{S}$ ; it is called space-like( Resp; time-like ,null) if every vector of  $T_p \mathcal{S}$  is contains space-like( resp. time-like ,null) vectors only.

**Definition 3.9.** *An immersion  $\sigma : U \rightarrow \mathcal{S} \in \mathbb{M}^{2,1}$  is called space-like , time-like, null if any tangent plane is space-like, time-like, null respectively.*

**Definition 3.10.** *The first fundamental form of a surface in  $\mathbb{M}^{2,1}$  is the expression*

$$Edu^2 + 2Fdudv + Gdv^2, \quad (3.25)$$

such that  $E = g(\sigma_u, \sigma_u)$ ,  $F = g(\sigma_u, \sigma_v)$  and  $G = g(\sigma_v, \sigma_v)$ .

Also, in tangent space written in terms of the basis  $\sigma_u, \sigma_v$  this can be represented in this basis by the symmetric matrix :

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}. \quad (3.26)$$

In Euclidean space, the first fundamental form is considered as a positive definite matrix. But in the Minkowskian case, the first fundamental form can be defined as in  $\mathbb{E}^3$  (3.26), while it is not necessarily positive definite.

Therefore; the first fundamental forms are classified by different types as follow:

**Definition 3.11.** [5] *A surface element  $\sigma : I \rightarrow \mathcal{S} \in \mathbb{M}^{2,1}$  is called :*

1. *space-like, in case the first fundamental form is positive definite,*
2. *time-like, in case the first fundamental form is indefinite, but non-degenerate.*
3. *null, in case the first fundamental form has rank 1.*

More specifically the first case corresponding to the case of Riemannian manifold. The second case is called the Minkowskian, here the first fundamental form has signature  $(-, +)$  everywhere on the surface. The third case is called null. The corresponding first fundamental form is degenerate.

In the same way, the normal unit vector, the second fundamental form and the surface curvatures; Gaussian and Mean curvature, all are defined by the same formula as in  $\mathbb{E}^3$ , just taking into account that the inner product and the vector product are to be understood in the sense of  $\mathbb{M}^{2,1}$  as defined above.

In this thesis, we will consider the second Minkowskian case. The first fundamental form has signature  $(-, +)$  everywhere. Thus, the surface is a Lorentzian manifold.

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This thesis is most concerned with the surfaces of rotation. In Minkowski space, there are three different types of matrix of rotations as well as three different types of surfaces. in te next chapter we will consider these types of rotations in more detail.



## 4. GENERALIZATION OF ROTATIONS IN 3D MINKOWSKI SPACE

Rotations in  $\mathbb{E}^3$  preserve all distance (i.e. they are isometries). As a consequence they also preserve all inner products, and map any orthonormal basis to another orthonormal basis; as with any linear transformation of finite dimensional vector spaces, a rotation can be represented by a matrix; once a basis is chosen.

Here we will first review the situation in  $\mathbb{E}^3$ , after that apply these concepts to  $\mathbb{M}^{2,1}$ .

### 4.1 Introduction

Recall the rotation matrix in three dimensional Euclidean space  $\mathbb{E}^3$ . Let  $R$  be a matrix of rotation, with standard basis  $e_x, e_y, e_z$  in  $\mathbb{E}^3$ . Since the standard basis is orthonormal, then

$$RIR^T = I, \tag{4.1}$$

where  $R^T$  is the transpose, and  $I$  is the identity matrix of Euclidean space. Furthermore, the rotation must preserve orientation, so we restrict ourselves to matrices  $R$ , with determinant 1.  $R$  is therefore an orthogonal matrix with unit determinant:

$$R^T = R^{-1} \quad \text{and} \quad \det R = 1. \tag{4.2}$$

By restricting attention to the proper rotation, we find that the set of all  $3 \times 3$  matrices which satisfies the property above (4.2) is given by  $SO(3)$ .

Here,  $SO(3)$  is a group with identity element of unit matrix  $I$  and the matrix multiplication as the group operation. We refer to  $SO(3)$  as the *rotation group* of  $\mathbb{E}^3$ .

Note that any  $R \in SO(3)$  has an eigenvector with eigenvalue of 1, which gives the axis of rotation.

For more information and details about these matrices see [23, 24].

Standard rotations in  $\mathbb{E}^3$  are those which fix one of the axes  $x, y$  or  $z$ ; they have simple representations.

$$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}, 0 \leq \theta \leq 2\pi. \quad (4.3)$$

This matrix describes the coordinate changes under a rotation, which is called rotation around the  $z$  axis. In other words, this rotation fixes the  $z$  axis.

Similarly, the matrix of rotation which fixes the  $y$  axis is :

$$R(\varphi) = \begin{pmatrix} \cos(\varphi) & 0 & \sin(\varphi) \\ 0 & 1 & 0 \\ -\sin(\varphi) & 0 & \cos(\varphi) \end{pmatrix}, 0 \leq \varphi \leq 2\pi, \quad (4.4)$$

and the matrix of rotation which fixes the  $x$  axis is:

$$R(\psi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\psi) & -\sin(\psi) \\ 0 & \sin(\psi) & \cos(\psi) \end{pmatrix}, 0 \leq \psi \leq 2\pi. \quad (4.5)$$

Note that, if  $R(\theta)$  is the rotation by  $\theta$  about some axis, and if  $\phi$  is another angle, then:

$$R(\theta)R(\phi) = R(\theta + \phi), \quad (4.6)$$

which means that, the group of matrices  $R(\theta)$  gives one parameter group of isometries.

The aim of this chapter is to generate an analogous group of rotational matrices in  $\mathbb{M}^{2,1}$ .

Firstly, we want to provide a general theory on isometries in pseudo-Riemannian manifolds, and introduce the Killing vector field. This helps in generating different type of matrices of rotation in  $\mathbb{M}^{2,1}$ .

## 4.2 The Isometry

We begin by considering the isometries in general in  $\mathbb{E}^3$ , then consider these for  $\mathbb{M}^{2,1}$ .

An *isometry* is a function that preserves the Riemannian metric.

**Definition 4.1.** A diffeomorphism  $\phi : (M, g) \rightarrow (N, h)$  is an isometry if  $\phi^*h = g$ .

This definition means that, for every point  $x$ ,  $d_x\phi$  is a linear isometry between  $T_xM$  and  $T_{\phi(x)}N$  see (3.24).

A particular consequence of this definition is that if  $f : M \rightarrow M$  is a diffeomorphism from a manifold onto itself, with the property that  $\forall p \in M$ , and all  $V, W \in T_pM$ ,

$$g_{f(p)}(f_p^*V, f_p^*W) = g_p(V, W), \quad (4.7)$$

then  $f$  is an isometry of  $(M, g)$ .

In terms of local coordinates, this leads:

$$g_{ab}(f(p)) \frac{\partial f^a}{\partial x^c} V^c \frac{\partial f^b}{\partial x^d} W^d = g_{cd}(p) V^c W^d, \quad (4.8)$$

and since this holds for all  $V, W$ , the condition for an isometry in local coordinates is

$$g_{ab}(f(p)) \frac{\partial f^a}{\partial x^c} \frac{\partial f^b}{\partial x^d} = g_{cd}(p). \quad (4.9)$$

A generic Riemannian manifold has no isometries other than the identity map.

The presence of an isometry is equivalent to the existence of a symmetry.

**Definition 4.2.** A one-parameter group of diffeomorphisms of a manifold  $\mathcal{M}$  is a smooth



map

$$\phi : \mathcal{M} \times \mathbb{R} \rightarrow \mathcal{M}, \quad (4.10)$$

such that  $\phi_t(x) = \phi(x, t)$ , where

- $\phi_t : \mathcal{M} \rightarrow \mathcal{M}$  is a diffeomorphism.
- $\phi_0 = id$ .
- $\phi_{s+t} = \phi_s \circ \phi_t$ .

This group is associated with a vector field  $V$  given by  $\frac{d}{dt}\phi_t(x) = V(x)$ , and the group of diffeomorphisms is called the flow of  $V$ .

If a one-parameter group of isometries is generated by a vector field  $V$ , then this vector field is called a **Killing vector field**.

### 4.3 Killing Vector Fields

To summarise we state that a Killing vector field or Killing vector is a vector field on a Riemannian manifold (or pseudo-Riemannian manifold) whose flow preserves the metric [37].

#### 4.3.1 Lie Derivative

Let us now recall the Lie derivative definition. This is a useful tool to interpret Killing fields.

**Definition 4.3.** Let  $V$  be a vector field on a smooth manifold  $M$  and  $\phi_t$  be the local flow generated by  $V$ . For each  $t \in \mathbb{R}$ , the map  $\phi_t$  is diffeomorphism of  $M$  and given a function  $f$  on  $M$ , we consider the Pull-back  $\phi_t^*f$ . We define the **Lie derivative** of the function  $f$  with respect to  $V$  by

$$L_V f = \lim_{t \rightarrow 0} \left( \frac{\phi_t^* f - f}{t} \right) = \left. \frac{d}{dt} \phi_t^* f \right|_{t=0}. \quad (4.11)$$

Let  $g_{ab}$  be any pseudo-Riemannian metric, then the Lie derivative is given by:

$$L_V g_{ab} = g_{ab,c} V^c + g_{ac} V_{,b}^c + g_{cb} V_{,a}^c. \quad (4.12)$$

In Cartesian coordinates in Euclidean and Minkowskian spaces where  $g_{ab,c} = 0$ , and the Lie derivative is given by:

$$L_V g_{ab} = g_{ac} V_{,b}^c + g_{cb} V_{,a}^c. \quad (4.13)$$

**Lemma 4.4.** [38] *The vector  $V$  generates a Killing field if and only if  $L_V g = 0$ .*

*Proof.* If  $V$  generate a Killing field, and  $\phi_t$  is a one parameter group of isometry generated by  $V$ , then  $\phi_t g = g$ . From (4.11), this yields  $L_V g = 0$ .  $\square$

If the Killing vector field has components  $V_a$ , then the condition that  $V$  be a Killing vector is that the covariant derivative of  $V$ , written  $V_{a;b}$  is skew symmetric [25].

### Properties of Killing fields

Some important properties of Killing fields are stated below

1. For any two Killing vector fields, their linear combination is also a Killing vector field.  
i.e  $aV + bW$  is KVF, and  $(a, b) \in \mathbb{R}$ .
2. The Lie bracket  $[U, V](f) = U(V(f)) - V(U(f)) = L_U V$  of two Killing vector fields  $U, V$  is also Killing vector field.
3. For a given Killing field  $V$ , and geodesic  $\gamma$  with velocity vector  $U$ , the quantity  $V_\mu U^\mu$  is constant along the geodesic  $\gamma$  [39].

4.3.2 Killing Fields of  $\mathbb{M}^{2,1}$ 

Let us consider the situation in  $\mathbb{M}^{2,1}$ , such that  $(\mathcal{M}, g) = (\mathbb{M}^{2,1}, \eta)$ , where  $\eta$  is the 3D Minkowski metric given by:

$$\eta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (4.14)$$

The Killing equation for any vector field  $V$  in Minkowski space is given by:

$$L_V \eta_{ab} = \eta_{ab,c} V^c + \eta_{ac} V_{,b}^c + \eta_{cb} V_{,a}^c = 0. \quad (4.15)$$

In Cartesian coordinates this is given by:

$$L_V \eta_{ab} = \eta_{ac} V_{,b}^c + \eta_{cb} V_{,a}^c = 0. \quad (4.16)$$

since  $\eta_{ab,c} = 0$

The general vector fields in  $\mathbb{M}^{2,1}$  are given by:

$$V := \xi(x, y, t) \frac{\partial}{\partial x} + \eta(x, y, t) \frac{\partial}{\partial y} + \tau(x, y, t) \frac{\partial}{\partial t}, \quad (4.17)$$

where  $\xi(x, y, t)$ ,  $\eta(x, y, t)$  and  $\tau(x, y, t)$  are real functions.

We seek for real functions  $\xi(x, y, t)$ ,  $\eta(x, y, t)$  and  $\tau(x, y, t)$  such that (4.17) is a Killing vector field.

In order to find the Killing vector of a given a metric  $\eta_{ij}$  we need to solve equation (4.16) which is a system of differential equations for the components of  $V$ . If (4.16) does not admit a solution the space-time has no symmetries. Note that although it may look like (4.16) is not covariant, since we are using the three dimensional coordinates in which  $\eta_{ab}$  are all constant, the equation is in fact equivalent to coordinate derivatives that are covariant

derivatives, and

$$g_{ac}V_{;b}^c + g_{cb}V_{;a}^c = 0. \quad (4.18)$$

Thus, from (4.18), we conclude that :

$$\xi_x = \eta_y = \tau_t = 0 \quad (4.19a)$$

$$\xi_y + \eta_x = \xi_t - \tau_x = \eta_t - \tau_y = 0. \quad (4.19b)$$

Now, from (4.19) we have

$$\xi_y + \eta_x = 0, \quad (4.20)$$

then differentiating with respect to  $x$  we get:

$$\xi_{yx} + \eta_{xx} = 0, \quad (4.21)$$

which gives  $\eta_{xx} = 0$ .

Therefore, the function  $\eta$  can be written as:

$$\eta(x, t) = f(t)x + g(t), \quad (4.22)$$

where  $f(t)$  and  $g(t)$  are functions of  $t$ .

Similarly, using the differentiation with respect to  $t$  for

$$\eta_t - \tau_y = 0 \quad (4.23)$$

we find  $\eta_{tt} = 0$ .

And from (4.22) we obtain

$$\eta_{tt} = f''(t)x + g''(t) = 0. \quad (4.24)$$

This gives  $f''(t) = g''(t) = 0$ . Thus,  $f'(t)$  and  $g'(t)$  are constant, so

$$f(t) = a_1 t + b_1, \quad \text{and} \quad g(t) = c_1 t + d_1, \quad (4.25)$$

such that all  $a_1, b_1, c_1$  and  $d_1$  are arbitrary constants.

Now, substituting (4.25) into (4.22) we have:

$$\eta = (a_1 t + b_1)x + c_1 t + d_1. \quad (4.26)$$

In the same way from (4.19), if we take the equations  $\xi_t - \tau_x = 0$  and  $\eta_t - \tau_y = 0$ , and differentiate with respect to  $x$  and  $t$  respectively; we get:

$$\tau_{xx} = \tau_{yy} = 0, \quad (4.27)$$

then, with the same calculation as above, one obtains:

$$\tau = (a_2 y + b_2)x + c_2 y + d_2, \quad (4.28)$$

where  $a_2, b_2, c_2$  and  $d_2$  are arbitrary constants.

Further, differentiation of  $\xi_y + \eta_x = 0$  and  $\xi_t + \tau_x$  with respect to  $y$  and  $t$  respectively; gives:

$$\xi_{yy} = \xi_{tt} = 0, \quad (4.29)$$

then, again as above, we obtain:

$$\xi = (a_3 t + b_3)y + c_3 t + d_3. \quad (4.30)$$

where  $a_3, b_3, c_3$  and  $d_3$  are arbitrary constants.

Now, substituting (4.26), (4.28) and (4.30) into (4.19b), yields:

$$\xi_y + \eta_x = 0 \implies a_3 t + b_3 + a_1 t + b_1 = 0. \quad (4.31)$$

Thus

$$\begin{aligned} a_1 + a_3 = 0 &\Rightarrow a_1 = -a_3 \\ b_1 + b_3 = 0 &\Rightarrow b_1 = -b_3. \end{aligned} \quad (4.32)$$

Similarly for the others, we can conclude that:

$$a_1 = a_2 = a_3 = 0 \quad , \text{and} \quad c_3 = b_2, c_1 = c_2. \quad (4.33)$$

In conclusion, the form of  $\xi, \eta$  and  $\tau$  can be given as follows:

$$\begin{aligned} \xi &= -b_1 y + b_2 t + d_3 \\ \eta &= b_1 x + c_1 t + d_1 \\ \tau &= b_2 x + c_1 y + d_2. \end{aligned} \quad (4.34)$$

Now substitute (4.34) into the general vector field equation (4.17), the Killing vector field can be written :

$$b_1 \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) + b_2 \left( x \frac{\partial}{\partial t} + t \frac{\partial}{\partial x} \right) + c_1 \left( y \frac{\partial}{\partial t} + t \frac{\partial}{\partial y} \right) + d_3 \frac{\partial}{\partial x} + d_1 \frac{\partial}{\partial y} + d_2 \frac{\partial}{\partial t} = 0. \quad (4.35)$$

This is the general solution of the Killing field equations in Minkowski space, which gives the full symmetry of special relativity ,including translations , rotations and boosts.

We are only interested in those transformations which fix some "axis of rotation" including the origin. Therefore, the coefficient of the translations will be set to zero. Thus we assume that  $d_1 = d_2 = d_3 = 0$ .

Then, the Killing vector field becomes:

$$V = \alpha \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) + \beta \left( x \frac{\partial}{\partial t} + t \frac{\partial}{\partial x} \right) + \gamma \left( y \frac{\partial}{\partial t} + t \frac{\partial}{\partial y} \right), \quad (4.36)$$

where  $\alpha, \beta, \gamma$  are constants.

By using the Killing equation (4.36), and using the conditions for the coordinates  $(x, y, t)$  we get the rotation matrices in  $\mathbb{M}^{2,1}$ . However, the Killing equation (4.36) can be parametrized in matrix form by the matrix  $L$  as follows:

$$V = L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & -\alpha & \beta \\ \alpha & 0 & \gamma \\ \beta & \gamma & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \quad (4.37)$$

Then any 1-parameter subgroup of  $SO(2, 1)$  is of the form  $e^{sL}$  for some  $L$  of this type.

Now, the determinant of the matrix  $L$  is 0 and the rank is 2, so  $L$  has one eigenvector with an eigenvalue of 0, and hence  $e^{sL}$  has an eigenvector with eigenvalue 1, i.e. the matrix of rotation leaves only one axis (line) fixed.

#### 4.4 Matrices of Rotation of $\mathbb{M}^{2,1}$

Be  $\Lambda$  a matrix of rotation, with standard basis  $e_x, e_y, e_t$  in  $\mathbb{M}^{2,1}$ . The rotation matrices are replaced by the Lorentz transformation  $\Lambda$ , such that,

$$\Lambda^T \eta \Lambda = \eta, \quad (4.38)$$

where  $\Lambda^T$  is the transpose, and  $\eta$  is metric matrix of Minkowski space.

The set of all  $3 \times 3$  matrices which satisfies the property (4.38) above is denoted by  $SO(2, 1)$ . The group of  $SO(2, 1)$  is a *group* under the operation of matrix multiplication.

In  $\mathbb{M}^{2,1}$ ; we have different types of axis of rotations and correspondingly different types of matrix of rotation; this section concerns these different types of matrices; and their dependence on the axis of rotation, time-like, space-like and light-like (null).

In this section we describe these matrices. i.e. we will find the subgroup of  $SO(2, 1)$  corresponding to rotations in  $\mathbb{E}^3$ .

Recall that, in  $\mathbb{E}^3$  all straight lines are equivalent, so all types of rotation are equivalent. But in  $\mathbb{M}^{2,1}$  there are three distinct types: time-like, space-like and null. Because of this, there are three types of 1-parameter subgroups of isometries of the Minkowski 3-dimensional space  $\mathbb{M}^{2,1}$  that leave a line (axis) pointwise fixed. We consider the rigid motion of the ambient space that makes the straight line fixed. So we investigate the corresponding rotation for each.

#### 4.4.1 Spatial Rotation in $\mathbb{M}^{2,1}$

Lets begin by seeking the most obvious analogue of rotation in  $\mathbb{M}^{2,1}$ : namely, we look for a one parameter group of Lorentz of transformations which fixes all points on the  $t$ -axis. This requires the Killing vector field to satisfy:

$$V(0, 0, t) = 0. \quad (4.39)$$

With applying equation (4.39) to (4.36); it results that,  $\beta = \gamma = 0$ , and  $\alpha$  can be any constant (say  $\alpha = 1$ ). As a result, the infinitesimal generator of this case is:

$$x\partial/\partial y - y\partial/\partial x. \quad (4.40)$$

Therefore, the Killing vector field becomes:

$$V = \begin{pmatrix} -y & x & 0 \end{pmatrix}^T. \quad (4.41)$$

We can now define a  $3 \times 3$  matrix  $M$  which corresponds to the infinitesimal generator, given in  $(x, y, t)$  coordinates by:

$$M = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (4.42)$$

The matrix  $M$  is the matrix corresponding to infinitesimal rotation about the  $t$  axis.



Now, we have the one- parameter group of homomorphisms  $\phi_s(x, y, t)$  given by:

$$\dot{\phi}_s(x) = M\phi_s(x). \quad (4.43)$$

So,

$$\phi_s(x) = e^{sM}x. \quad (4.44)$$

Calculating the matrix exponential gives:

$$\begin{pmatrix} \cos s & -\sin s & 0 \\ \sin s & \cos s & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (4.45)$$

Therefore, the one parameter subgroup of rotation matrices is:

$$\Lambda_t(s) = \begin{pmatrix} \cos s & -\sin s & 0 \\ \sin s & \cos s & 0 \\ 0 & 0 & 1 \end{pmatrix}, -\infty < s < \infty. \quad (4.46)$$

That means, however in conclusion; any point  $(0, 0, t)$  is fixed. Thus the axis of rotation is given by  $l = (0, 0, 1)$ . Also the orbit of any point of a space-like curve with  $t$ - constant, in this case is a circle centred at the origin.

So, if the point  $p$  has coordinates  $(x, y, t)$ , then the orbit is given by :

$$(x \cos s - y \sin s, x \sin s + y \cos s, t) \quad (4.47)$$

Obviously the  $t$  coordinate is fixed.

#### 4.4.2 Boost in Direction of Space-like Axis in $\mathbb{M}^{2,1}$

In this section the one parameter group of transformations which fixes each point in a space-like line is sought. Let  $y$ - axis be the axis of rotation.

Then we must have:

$$V(0, y, 0) = 0, \quad (4.48)$$

for all  $y$ .

Applying equation (4.48) to (4.36); yields,  $\alpha = \gamma = 0$ , and  $\beta$  can be any constant, (say  $\beta = 1$ ). As a result, the infinitesimal generator of this case is:

$$x\partial/\partial t + t\partial/\partial x. \quad (4.49)$$

Therefore, (by the same argument as in spatial rotation (Sec 4.4.1)), the one parameter subgroup of rotation matrices consists of:

$$\Lambda_y(s) = \begin{pmatrix} \cosh(s) & 0 & \sinh(s) \\ 0 & 1 & 0 \\ \sinh(s) & 0 & \cosh(s) \end{pmatrix}, -\infty < s < \infty. \quad (4.50)$$

And the axis of rotation is the  $y$ -axis.

In this case, the orbit of any point has fixed  $y$ -coordinate, and it is a hyperbola-timelike if  $|x_0| < |t_0|$ , spacelike if  $|x_0| > |t_0|$ , and degenerate to a null if  $|x_0| = |t_0|$ .

So, let  $p = (x, y, t)$ , then the orbit here is given by:

$$(x \cosh s + t \sinh s, y, x \sinh s + t \cosh s), \quad (4.51)$$

obviously, the  $y$  coordinate is fixed.

#### 4.4.3 Null Rotation in $\mathbb{M}^{2,1}$

Finally, consider the situation where the axis of rotation is a null line. Say it is located in  $yt$ -plane, say  $y = t$ . Then

$$V(0, s, s) = 0, \quad (4.52)$$

must vanish for all  $s$ .

Applying equation (4.52) to (4.36) results in  $\alpha = \beta$  being any constants (say  $\alpha = \beta = 1$ ) and  $\gamma = 0$ . Consequently, the infinitesimal generator of this case is:

$$(y - t)\partial/\partial x + x\partial/\partial y + x\partial/\partial t. \quad (4.53)$$

Therefore, the Killing vector field is:

$$V = \begin{pmatrix} t - y \\ x \\ x \end{pmatrix}. \quad (4.54)$$

Then the  $3 \times 3$  matrix  $M$  which corresponds to the infinitesimal generator, can be given in  $(x, y, t)$  coordinates by:

$$M = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (4.55)$$

The matrix  $M$  is called the infinitesimal matrix corresponding to rotation about the null axis.

As before, the one parameter subgroup of rotation matrices consists of:

$$\begin{pmatrix} 1 & -s & s \\ s & 1 - \frac{s^2}{2} & \frac{s^2}{2} \\ s & -\frac{s^2}{2} & 1 + \frac{s^2}{2} \end{pmatrix}. \quad (4.56)$$

This one-parameter group fixes the line  $y = t$  in  $yt$  plane. i.e. If the point  $q = (0, v, v)$  is a

point on the line  $y = t$  then ,

$$\begin{pmatrix} 1 & -s & s \\ s & 1 - \frac{s^2}{2} & \frac{s^2}{2} \\ s & -\frac{s^2}{2} & 1 + \frac{s^2}{2} \end{pmatrix} \begin{pmatrix} 0 \\ v \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ v \\ v \end{pmatrix}. \quad (4.57)$$

Therefore, the one parameter subgroup of rotation matrices is:

$$\Lambda(s) = \begin{pmatrix} 1 & -s & s \\ s & 1 - \frac{s^2}{2} & \frac{s^2}{2} \\ s & -\frac{s^2}{2} & 1 + \frac{s^2}{2} \end{pmatrix}, -\infty < s < \infty. \quad (4.58)$$

So the axis of rotation is given by  $l = (0, 1, 1)$  . And the orbit of any point has a fix line  $y = t$ , and the orbit is parabola-time-like. So,if  $p$  has the coordinate  $(x, y, t)$  , then the orbit is given by :

$$\begin{pmatrix} x - sy + st \\ sx + (1 - 1/2 s^2) y + 1/2 s^2 t \\ sx - 1/2 s^2 y + (1 + 1/2 s^2) t \end{pmatrix}. \quad (4.59)$$

In conclusion the matrices (4.46),(4.50) and (4.58) are called elliptic , (resp: hyperbolic and parabolic) 1-parameter subgroup. A surface is called *surface of revolution*  $\mathcal{S}$  if its image is stable under a one parameter group of isometries which leave a line pointwise fixed. This general definition will be related to the ordinary one in terms of rotating a curve which lies in a certain plane containing the axis of rotation [21]. Our plan is to explore various types of surfaces of rotation, that are generated by these matrices.



## 5. GENERALIZING CLAIRAUT'S THEOREM TO 3D MINKOWSKI SPACE

A surface of rotation in Euclidean space is generated by rotating an arbitrary curve about an arbitrary axis, (Sec2.2). In Minkowski space, however, there are different types of curves as well as different types of rotation axes (time-like, space-like and null), so that there are different types of surfaces of rotation.

This chapter will explore these different types of surfaces of rotation, and will generalize Clairaut's theorem to these surfaces.

### 5.1 Surfaces of Rotation in 3D Minkowski Space

**Definition 5.1.** *A surface  $\mathcal{S}$  in  $\mathbb{M}^{2,1}$  is a surface of revolution or rotational surface, if  $\mathcal{S}$  is invariant under a one-parameter subgroup of isometries which leave a line pointwise fixed.*

This general definition will be related to the ordinary one in terms of rotating a profile curve lying in a certain plane containing the axis of rotation (See [21],[22]).

#### 5.1.1 Surfaces of Rotation Generated by Time-like Rotation

By time-like rotation we mean; rotating a curve using the spatial rotation matrix (4.46), i.e. the axis of rotation is given by the eigen vector  $l = (0, 0, 1)$ . Clearly, any point in  $\mathbb{M}^{2,1}$  can be carried to the  $xt$ - plane by some notation, so without loss of generality we assume that the curve  $\gamma$  lies in the  $xt$ - plane. Hence, one of its parametrizations is:

$$\gamma(u) = (\rho(u), 0, h(u)), \quad (5.1)$$

where,  $\rho(u), h(u)$  are smooth functions, and we assume that  $\rho(u)$  is a positive function.

Hence, the surface of revolution  $\mathcal{S}^t$  around  $t$  can be parametrized as :

$$\mathcal{S}^t(u, v) = \begin{pmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \rho(u) \\ 0 \\ h(u) \end{pmatrix}. \quad (5.2)$$

So,

$$\mathcal{S}^t(u, v) = (\rho(u) \cos v, \rho(u) \sin v, h(u)), u \in I, 0 \leq v \leq 2\pi. \quad (5.3)$$

This rotation is still within the Euclidean plane.

Furthermore,

$$\mathcal{S}_u^t = \begin{pmatrix} \rho'(u) \cos(v) \\ \rho'(u) \sin(v) \\ h'(u) \end{pmatrix} \quad \text{and} \quad \mathcal{S}_v^t = \begin{pmatrix} -\rho(u) \sin(v) \\ \rho(u) \cos(v) \\ 0 \end{pmatrix}, \quad (5.4)$$

so,

$$\begin{cases} E = g(\mathcal{S}_u^t, \mathcal{S}_u^t) = \rho'^2(u) - h'^2(u) \\ F = g(\mathcal{S}_u^t, \mathcal{S}_v^t) = 0 \\ G = g(\mathcal{S}_v^t, \mathcal{S}_v^t) = \rho^2(u), \end{cases} \quad (5.5)$$

We assume that the functions  $\rho(u), h(u)$  have the property of  $\rho'^2(u) - h'^2(u) = -1$ . So the curve  $\gamma$  is time-like and parametrized by a proper time, this results in the first fundamental form

$$I_{\mathcal{S}^t} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & \rho^2(u) \end{pmatrix}. \quad (5.6)$$

Thus the first fundamental form of  $\mathcal{S}^t(u, v)$  has signature  $(-, +)$  everywhere, which gives a Lorentz metric on  $\mathcal{S}^t$ .

Also,

$$\mathcal{S}_{uu}^t = \begin{pmatrix} \rho''(u) \cos(v) \\ \rho''(u) \sin(v) \\ h''(u) \end{pmatrix}, \quad \mathcal{S}_{uv}^t = \begin{pmatrix} -\rho'(u) \sin(v) \\ \rho'(u) \cos(v) \\ 0 \end{pmatrix} \quad \text{and} \quad \mathcal{S}_{vv}^t = \begin{pmatrix} -\rho(u) \cos(v) \\ -\rho(u) \sin(v) \\ 0 \end{pmatrix}. \quad (5.7)$$

Moreover, the unit normal vector (2.7) is

$$n_{\mathcal{S}^t} = \begin{pmatrix} h'(u) \cos(v) \\ h'(u) \sin(v) \\ \rho'(u) \end{pmatrix}. \quad (5.8)$$

So,

$$\begin{cases} L = g(\mathcal{S}_{uu}^t, n_{\mathcal{S}^t}) = \rho''(u)h'(u) - \rho'(u)h''(u) \\ M = g(\mathcal{S}_{uv}^t, n_{\mathcal{S}^t}) = 0 \\ N = g(\mathcal{S}_{vv}^t, n_{\mathcal{S}^t}) = -\rho(u)h'(u), \end{cases} \quad (5.9)$$

This results in the second fundamental form of :

$$\mathbb{I}_{\mathcal{S}^t} = \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} \rho''(u)h'(u) - \rho'(u)h''(u) & 0 \\ 0 & -\rho(u)h'(u) \end{pmatrix}. \quad (5.10)$$

Therefore, the Gaussian (2.10), and mean curvatures (2.11) are given by:

$$K = -\rho''(u)/\rho(u) \quad \text{and} \quad H = \frac{1}{2} (\rho''(u)h'(u) - \rho'(u)h''(u) - h'(u)/\rho(u)) = \frac{1}{2} \left( \frac{\rho''(u)}{h'(u)} - \frac{h'(u)}{\rho(u)} \right). \quad (5.11)$$

And again note that, as with the Frenet-Serret equation, these are different from the Euclidean case in sign.

Furthermore, the condition of constant mean curvature CMC of this family of surfaces



of revolution is that,

$$\frac{1}{2} \left( \frac{h'^2(u) - \rho(u)\rho''(u)}{\rho(u)h'(u)} \right) = C, \quad (5.12)$$

where  $C$  is constant.

One can consider the special case of CMC surfaces defined by a zero mean curvature which are *minimal surfaces* in the Euclidean case. So the condition of "minimal surfaces" of this family of surfaces is

$$\rho(u)\rho''(u) = 1 + \rho'^2(u). \quad (5.13)$$

The equation above (5.13) can be solved numerically as a differential equation to find the minimal surface. However [18] classified the surfaces of rotation of this case to two groups of surfaces, one group being surfaces of revolutions and the other called circular of the cylinder.

There are also many surfaces of constant mean curvature of this family. (See [18], [22], [26], [27]).

### 5.1.2 Surfaces of Rotation Generated by a Boost in the Direction of Space(Time)-like Axes

This surface of revolution is generated by "rotating" a curve using a boost in the  $x$ -axis direction (4.50), i.e. the axis of rotation is given by the eigenvector  $l = (0, 1, 0)$ .

In the case of a boost, we have two cases of surfaces of revolution, depending on the generating curve. Since, for any point  $\{x, y, t\}$  in  $\mathbb{M}^{2,1}$  as in chapter four (4.4.2) the orbit is hyperbola-time-like if  $|x_0| > |t_0|$ , and space-like if  $|x_0| < |t_0|$ . Then we will have two cases of the curve  $\gamma$ . This means the curve  $\gamma$  lies in the  $xy$ -plane and has  $|x| > |t|$  or  $yt$ -plane with  $|x| < |t|$ . So, the curve is either space-like parametrized or time-like parametrized. Therefore we have two surfaces of revolution as follows:

#### *Surfaces of Rotation Generated by a Boost with a Space-like Parametrized Curve*

In this case, the curve  $\gamma$  lies in the  $xy$ -plane. Hence, one of its parametrizations is:

$$\gamma(u) = (\rho(u), h(u), 0), \quad (5.14)$$

where,  $\rho(u), h(u)$  are smooth functions.

However, the surface of revolution  $\mathcal{S}^{xy}$  around  $y$  can be parametrized as :

$$\mathcal{S}^{xy}(u, v) = \begin{pmatrix} \cosh(v) & 0 & \sinh(v) \\ 0 & 1 & 0 \\ \sinh(v) & 0 & \cosh(v) \end{pmatrix} \begin{pmatrix} \rho(u) \\ h(u) \\ 0 \end{pmatrix}. \quad (5.15)$$

So,

$$\mathcal{S}^{xy}(u, v) = (\rho(u) \cosh v, h(u), \rho(u) \sinh v), u \in I, -\infty < v < \infty, \quad (5.16)$$

and we see that the orbit of each point on the generating curve is space-like.

Moreover,

$$\mathcal{S}_u^{xy} = \begin{pmatrix} \rho'(u) \cosh(v) \\ h'(u) \\ \rho'(u) \sinh(v) \end{pmatrix}, \quad \text{and} \quad \mathcal{S}_v^{xy} = \begin{pmatrix} \rho(u) \sinh(v) \\ 0 \\ \rho(u) \cosh(v) \end{pmatrix}, \quad (5.17)$$

So,

$$\begin{cases} E = g(\mathcal{S}_u^{xy}, \mathcal{S}_u^{xy}) = \rho'^2(u) + h'^2(u) \\ F = g(\mathcal{S}_u^{xy}, \mathcal{S}_v^{xy}) = 0 \\ G = g(\mathcal{S}_v^{xy}, \mathcal{S}_v^{xy}) = -\rho^2(u), \end{cases} \quad (5.18)$$

In this case the curve  $\gamma$  is space-like curve. Or the curve  $\gamma$  can be parametrized by space-like definite, i.e.  $\rho'^2(u) + h'^2(u) = 1$ , then the first fundamental form of this surface is given by:

$$I_{\mathcal{S}^{xy}} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -\rho^2(u) \end{pmatrix}. \quad (5.19)$$

Thus the first fundamental form has signature  $(+, -)$  everywhere.

Similarly, as in previous section, the coefficients of the second fundamental form are

given by:

$$\begin{cases} L = g(\mathcal{S}_{uu}^{xy}, n_{\mathcal{S}^{xy}}) = \rho'(u)h''(u) - \rho''(u)h'(u) \\ M = g(\mathcal{S}_{uv}^{xy}, n_{\mathcal{S}^{xy}}) = 0 \\ N = g(\mathcal{S}_{vv}^{xy}, n_{\mathcal{S}^{xy}}) = -\rho(u)h'(u). \end{cases} \quad (5.20)$$

So, the second fundamental form is

$$\mathbb{I}_{\mathcal{S}^{xy}} = \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} \rho'(u)h''(u) - \rho''(u)h'(u) & 0 \\ 0 & -\rho(u)h'(u) \end{pmatrix}. \quad (5.21)$$

Therefore, the Gaussian and mean curvatures (2.10),(2.11) are given by:

$$K = -\rho''(u)/\rho(u) \quad \text{and} \quad H = \frac{1}{2} (\rho'(u)h''(u) - \rho''(u)h'(u) + h'(u)/\rho(u)) = \frac{1}{2} \left( -\frac{\rho''(u)}{h'(u)} + \frac{h'(u)}{\rho(u)} \right), \quad (5.22)$$

Furthermore, the condition of constant mean curvature CMC of this family of surfaces of revolution is that,

$$\frac{1}{2} \left( \frac{h'^2(u) - \rho(u)\rho''(u)}{\rho(u)h'(u)} \right) = C, \quad (5.23)$$

where  $C$  is constant.

One can think of special cases of CMC surfaces defined by a zero mean curvature which are minimal surfaces in Euclidean case. So the condition for minimal surfaces of this family of surfaces is

$$\rho(u)\rho''(u) = 1 - \rho'^2(u). \quad (5.24)$$

The classification of surfaces of rotation in this case are either surfaces of revolution or Lorentz hyperbolic cylinders or minimal [18].

For example , there is a minimal surface given explicitly in [18] That is, the surface of rotation which is parametrized by:

$$\mathcal{S}^{xy}(u, v) = ((u + c_1) \cosh v, c_2, (u + c_1) \sinh v), \quad (5.25)$$

where,  $c_1, c_2 \in \mathbb{R}$ .

On the other hand there are many surfaces of constant mean curvature of this family of surfaces. (See [18], [22], [26], [27]).

### *Surfaces of Rotation Generated by a Boost with a Time-like Parametrized Curve*

In this case, the curve  $\gamma$  lies in the  $yt$ - plane. Hence, it is parametrized by:

$$\gamma(u) = (0, h(u), \rho(u)), \quad (5.26)$$

where,  $\rho(u), h(u)$  are smooth functions.

However, the surface of revolution  $\mathcal{S}^{yt}$  around  $y$  can be parametrized as :

$$\mathcal{S}^{yt}(u, v) = \begin{pmatrix} \cosh(v) & 0 & \sinh(v) \\ 0 & 1 & 0 \\ \sinh(v) & 0 & \cosh(v) \end{pmatrix} \begin{pmatrix} 0 \\ h(u) \\ \rho(u) \end{pmatrix}. \quad (5.27)$$

So,

$$\mathcal{S}^{yt}(u, v) = (\rho(u) \sinh v, h(u), \rho(u) \cosh v), u \in I, -\infty < v < \infty. \quad (5.28)$$

This time the orbit of each point on the generating curve is time-like.

Moreover,

$$\mathcal{S}_u^{yt} = \begin{pmatrix} \rho'(u) \sinh(v) \\ h'(u) \\ \rho'(u) \cosh(v) \end{pmatrix}, \quad \text{and} \quad \mathcal{S}_v^{yt} = \begin{pmatrix} \rho(u) \cosh(v) \\ 0 \\ \rho(u) \sinh(v) \end{pmatrix}, \quad (5.29)$$

So, we have:

$$\begin{cases} E = g(\mathcal{S}_u^{yt}, \mathcal{S}_u^{yt}) = h'^2(u) - \rho'^2(u) = -1 \\ F = g(\mathcal{S}_u^{yt}, \mathcal{S}_v^{yt}) = 0 \\ G = g(\mathcal{S}_v^{yt}, \mathcal{S}_v^{yt}) = \rho^2(u). \end{cases} \quad (5.30)$$

The first fundamental form of this surface is given by:

$$I_{\mathcal{S}^{yt}} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & \rho^2(u) \end{pmatrix}. \quad (5.31)$$

Thus the first fundamental form has signature  $(-, +)$  everywhere.

Similarly, as in the previous section, the coefficients of the second fundamental form are given by:

$$\begin{cases} L = g(\mathcal{S}_{uu}^{yt}, n_{\mathcal{S}^{yt}}) = \rho''(u)h'(u) - \rho'(u)h''(u) \\ M = g(\mathcal{S}_{uv}^{yt}, n_{\mathcal{S}^{yt}}) = 0 \\ N = g(\mathcal{S}_{vv}^{yt}, n_{\mathcal{S}^{yt}}) = \rho(u)h'(u). \end{cases} \quad (5.32)$$

So, the second fundamental form is

$$\mathbb{I}_{\mathcal{S}^{yt}} = \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} \rho''(u)h'(u) - \rho'(u)h''(u) & 0 \\ 0 & \rho(u)h'(u) \end{pmatrix}. \quad (5.33)$$

Therefore, the Gaussian and mean curvatures are given by:

$$K = -\rho''(u)/\rho(u) \quad \text{and} \quad H = \frac{1}{2} (\rho'(u)h''(u) - \rho''(u)h'(u) + h'(u)/\rho(u)) = \frac{1}{2} \left( \frac{-\rho''(u)}{h'(u)} + \frac{h'(u)}{\rho(u)} \right). \quad (5.34)$$

Furthermore, the condition of constant mean curvature CMC of this family of surfaces of revolution is that,

$$\frac{1}{2} \left( \frac{h'^2(u) - \rho(u)\rho''(u)}{\rho(u)h'(u)} \right) = C, \quad (5.35)$$

where  $C$  is constant.

One can think of the special case of CMC surfaces defined by a zero mean curvature and which are minimal surfaces in the Euclidean case. So the condition of minimal surfaces of this family of surfaces is

$$\rho(u)\rho''(u) = \rho'^2(u) - 1. \quad (5.36)$$

### 5.1.3 Surfaces of Rotation Generated by Null Rotation

By the null rotation we mean rotating a curve using null rotation matrix (4.58), i.e. the axis of rotation is given by  $l = (0, 1, 1)$ .

For any choice of point  $p = (x, y, t)$  the orbit in this case is given in (4.59) by:

$$\begin{pmatrix} x - v(y + t) \\ vx + (1 - 1/2 v^2) y + 1/2 v^2 t \\ vx - 1/2 v^2 y + (1 + 1/2 v^2) t \end{pmatrix}. \quad (5.37)$$

So, if  $y \neq t$  then there exist  $v$  such that:

$$\begin{pmatrix} x - v(y + t) \\ vx + (1 - 1/2 v^2) y + 1/2 v^2 t \\ vx - 1/2 v^2 y + (1 + 1/2 v^2) t \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{y} \\ \tilde{t} \end{pmatrix}. \quad (5.38)$$

So, without loss of generality, we assume that the curve  $\gamma$  lies in the  $yt$ - plane. Hence, one of its parametrizations is:

$$\gamma(u) = (0, q(u), h(u)), \quad (5.39)$$

where,  $q(u), h(u)$  are smooth functions, and  $q(u) - h(u) \neq 0$ .

Hence, the surface of revolution  $\mathcal{S}^n$  around the line  $y = t$  can be parametrized as :

$$\mathcal{S}^n(u, v) = \begin{pmatrix} 1 & -v & v \\ v & 1 - \frac{v^2}{2} & \frac{v^2}{2} \\ v & -\frac{v^2}{2} & 1 + \frac{v^2}{2} \end{pmatrix} \begin{pmatrix} 0 \\ q(u) \\ h(u) \end{pmatrix}. \quad (5.40)$$

So,

$$\mathcal{S}^n(u, v) = \begin{pmatrix} -vq(u) + vh(u) \\ (1 - \frac{v^2}{2})q(u) + \frac{v^2}{2}h(u) \\ -\frac{v^2}{2}q(u) + (1 + \frac{v^2}{2})h(u) \end{pmatrix}, u \in I, -\infty < v < \infty. \quad (5.41)$$

In addition,

$$\mathcal{S}_u^n = \begin{pmatrix} -vq'(u) + vh'(u) \\ (1 - \frac{v^2}{2})q'(u) + \frac{v^2}{2}h'(u) \\ -\frac{v^2}{2}q'(u) + (1 + \frac{v^2}{2})h'(u) \end{pmatrix}, \quad (5.42)$$

also

$$\mathcal{S}_v^n = \begin{pmatrix} -q(u) + h(u) \\ -vq(u) + vh(u) \\ -vq(u) + vh(u) \end{pmatrix}. \quad (5.43)$$

So,

$$\begin{cases} E = g(\mathcal{S}_u^n, \mathcal{S}_u^n) = \rho'^2(u) - h'^2(u) \\ F = g(\mathcal{S}_u^n, \mathcal{S}_v^n) = 0 \\ G = g(\mathcal{S}_v^n, \mathcal{S}_v^n) = (q(u) - h(u))^2 = \rho^2(u). \end{cases} \quad (5.44)$$

Then taking that the functions  $q(u), h(u)$  have the property  $q'^2(u) - h'^2(u) = -1$ , the curve  $\gamma$  is time-like and parametrized by a proper time, and given by  $\rho(u) = q(u) - h(u)$  which measures the distance from the axis of rotation.

So, the first fundamental form of this surface is given by:

$$I_{\mathcal{S}^n} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & \rho^2(u) \end{pmatrix}. \quad (5.45)$$

The first fundamental form has signature  $(-, +)$  everywhere.

Similarly, as before, the coefficients of the second fundamental form is given by:

$$\begin{cases} L = g(\mathcal{S}_{uu}^n, n_{\mathcal{S}^n}) = q''(u)h'(u) - q'(u)h''(u) \\ M = g(\mathcal{S}_{uv}^n, n_{\mathcal{S}^n}) = 0 \\ N = g(\mathcal{S}_{vv}^n, n_{\mathcal{S}^n}) = (q(u) - h(u))(q'(u) - h'(u)). \end{cases} \quad (5.46)$$

So, the second fundamental form is given by:

$$\mathbb{I}_{\mathcal{S}^n} = \begin{pmatrix} q''(u)h'(u) - q'(u)h''(u) & 0 \\ 0 & (q(u) - h(u))(q'(u) - h'(u)) \end{pmatrix}. \quad (5.47)$$

Therefore, the Gaussian and mean curvatures are given by:

$$K = \frac{q'(u) - h'(u)}{q(u) - h(u)} (h''(u)q'(u) - q''(u)h'(u)), \quad (5.48)$$

and,

$$H = \frac{1}{2} \left( q'(u)h''(u) - q''(u)h'(u) + \frac{q'(u) - h'(u)}{q(u) - h(u)} \right) = \frac{1}{2} \left( \frac{-q''(u)}{h'(u)} + \frac{q'(u) - h'(u)}{q(u) - h(u)} \right). \quad (5.49)$$

This equation (5.49) may be solved numerically to find either CMC or minimal surface. But we are interested in generalizing Clairaut's theorem to this surface more than the properties of the surface itself.

## 5.2 Clairaut's Theorem in 3D Minkowski Space

In this section, we will generalize Clairaut's theorem to the four surfaces of rotations above (Sec(5.1)). We will find that in each case we have Clairaut's theorem in Minkowski space.

### 5.2.1 Clairaut's Theorem of Surface of Rotation Generated by Time-like Rotation

The surface of rotation in this case is parametrized by:

$$\mathcal{S}^t(u, v) = (\rho(u) \cos v, \rho(u) \sin v, h(u)), \quad (5.50)$$

and the functions  $\rho(u), h(u)$  have the property of  $\rho'^2(u) - h'^2(u) = -1$ , so the curve  $\gamma$  is time-like and parametrized by a proper time.



Recall,

$$\mathcal{S}_u^t = \begin{pmatrix} \rho'(u) \cos(v) \\ \rho'(u) \sin(v) \\ h'(u) \end{pmatrix} = \mathbf{n}_u \quad \text{and} \quad \mathcal{S}_v^t = \begin{pmatrix} -\rho(u) \sin(v) \\ \rho(u) \cos(v) \\ 0 \end{pmatrix} = \rho(u) \mathbf{n}_v, \quad (5.51)$$

resulting in the first fundamental form

$$I = \begin{pmatrix} -1 & 0 \\ 0 & \rho^2(u) \end{pmatrix}. \quad (5.52)$$

We note that  $\mathcal{S}_u^t = \mathbf{n}_u$  is a unit time-like vector pointing along the meridians, while  $\mathcal{S}_v^t = \rho \mathbf{n}_v$ , such that  $\mathbf{n}_v$  is a unit space-like vector pointing along the parallels. As in the Euclidean case,  $g(\mathbf{n}_u, \mathbf{n}_v) = 0$ . So we have an orthonormal basis, and hence a unit time-like vector  $t$  tangent to  $\mathcal{S}^t$  can be written  $\mathbf{n}_u \cosh \psi + \mathbf{n}_v \sinh \psi$  where  $\psi$  is the hyperbolic angle between  $t$  and  $\mathbf{n}_u$ .

This time the Lagrangian is

$$- \dot{u}^2 + \rho^2 \dot{v}^2, \quad (5.53)$$

giving Euler-Lagrange equations,

$$\begin{aligned} \ddot{u} &= -\rho \rho' \dot{v}^2 \\ \frac{d}{ds} (\rho^2 \dot{v}) &= 0. \end{aligned} \quad (5.54)$$

Now let  $\gamma$  be a timelike geodesic on  $\mathcal{S}^t$ , given by  $u(s), v(s)$ . Then as before (Sec. (2.26)), we have:

$$\begin{aligned} \dot{\gamma} &= \dot{u} \mathcal{S}_u^t + \dot{v} \mathcal{S}_v^t \\ &= \dot{u} \mathbf{n}_u + \rho \dot{v} \mathbf{n}_v. \end{aligned} \quad (5.55)$$

In the Minkowski setting, however, this gives:

$$\dot{\boldsymbol{\gamma}} = \mathbf{n}_u \cosh \psi + \mathbf{n}_v \sinh \psi \quad (5.56)$$

where  $\psi$  is now the hyperbolic angle between  $\dot{\boldsymbol{\gamma}}$  and  $\mathbf{n}_u$ , i.e. between  $\dot{\boldsymbol{\gamma}}$  and a meridian.

We then see that the second Euler-Lagrange equation is equivalent to  $\rho \sinh \psi$  being constant.

Conversely, let  $\boldsymbol{\gamma}$  be a proper-time parametrized curve such that  $\rho \sinh \psi = \rho^2 \dot{v}$  is constant, and  $\dot{u} \neq 0$ . We then have:

$$\dot{u}^2 - \rho^2 \dot{v}^2 = 1 \quad \text{and} \quad \rho^2 \dot{v} = \text{constant}$$

Differentiating this gives:

$$\begin{aligned} \dot{u}\ddot{u} - \rho\rho'\dot{u}\dot{v}^2 - \rho^2\dot{v}\ddot{v} &= 0 \\ 2\rho\rho'\dot{u}\dot{v} + \rho^2\ddot{v} &= 0 \end{aligned} \quad (5.57)$$

Multiplying the second equation by  $\dot{v}$  and substituting into the first gives

$$\dot{u}\ddot{u} + \rho\rho'\dot{u}\dot{v}^2 = 0 \quad (5.58)$$

and since  $\dot{u} \neq 0$  we have

$$\ddot{u} = -\rho\rho'\dot{v}^2 \quad (5.59)$$

which is the first Euler-Lagrange equation. It follows that  $\boldsymbol{\gamma}$  is a time-like geodesic.

□

### 5.2.2 Clairaut's Theorem of Surface of Rotation Generated by a Boost

**First we consider the case of space-like generators:**

The surface of rotation in this case is parametrized by:

$$\mathcal{S}^{xy}(u, v) = (\rho(u) \cosh v, h(u), \rho(u) \sinh v), \quad (5.60)$$

and the functions  $\rho(u), h(u)$  have the property of  $\rho'^2(u) + h'^2(u) = 1$ .

Furthermore,

$$\mathcal{S}_u^{xy} = \begin{pmatrix} \rho'(u) \cosh(v) \\ h'(u) \\ \rho'(u) \sinh(v) \end{pmatrix} = \mathbf{n}_u \quad \text{and} \quad \mathcal{S}_v^{xy} = \begin{pmatrix} \rho(u) \sinh(v) \\ 0 \\ \rho(u) \cosh(v) \end{pmatrix} = \rho(u) \mathbf{n}_v, \quad (5.61)$$

resulting in the first fundamental form :

$$I = \begin{pmatrix} 1 & 0 \\ 0 & -\rho^2(u) \end{pmatrix}. \quad (5.62)$$

We note that  $\mathcal{S}_u^{xy} = \mathbf{n}_u$  is a unit space-like vector pointing along the meridians, while  $\mathcal{S}_v^{xy} = \rho \mathbf{n}_v$ , such that  $\mathbf{n}_v$  is a unit time-like vector pointing along the parallels. And,  $g(\mathbf{n}_u, \mathbf{n}_v) = 0$ . So we have an orthonormal basis, and hence a unit time-like vector  $t$  tangent to  $\mathcal{S}^{xy}$  can be written  $\mathbf{n}_u \cosh \psi + \mathbf{n}_v \sinh \psi$  where  $\psi$  is the hyperbolic angle between  $t$  and  $\mathbf{n}_u$ .

This time the Lagrangian is

$$\dot{u}^2 - \rho^2 \dot{v}^2 \quad (5.63)$$

giving Euler-Lagrange equations,

$$\begin{aligned} \ddot{u} &= -\rho \rho' \dot{v}^2 \\ \frac{d}{ds} (\rho^2 \dot{v}) &= 0. \end{aligned} \quad (5.64)$$

Now let  $\gamma$  be a time-like geodesic on  $\mathcal{S}^{xy}$ , given by  $u(s), v(s)$ . Then we have

$$\begin{aligned}\dot{\gamma} &= \dot{u}\mathcal{S}_u^{xy} + \dot{v}\mathcal{S}_v^{xy} \\ &= \dot{u}\mathbf{n}_u + \rho\dot{v}\mathbf{n}_v.\end{aligned}\tag{5.65}$$

This gives

$$\dot{\gamma} = \mathbf{n}_u \cosh \psi + \mathbf{n}_v \sinh \psi \tag{5.66}$$

where  $\psi$  is the hyperbolic angle between  $\dot{\gamma}$  and  $\mathbf{n}_u$ , i.e. between  $\dot{\gamma}$  and a meridian.

We then see that the second Euler-Lagrange equation is again equivalent to  $\rho \sinh \psi$  being constant.

Conversely, let  $\gamma$  be a proper-time parametrized curve such that  $\rho \sinh \psi = \rho^2 \dot{v}$  is constant, and  $\dot{u} \neq 0$ . We then have

$$\dot{u}^2 - \rho^2 \dot{v}^2 = 1 \quad \text{and} \quad \rho^2 \dot{v} = \text{constant}$$

This calculation is identical to previous case.

It follows that  $\gamma$  is a time-like geodesic.

□

### Now we consider the case of time-like generator

The surface of rotation in this case is parametrized by:

$$\mathcal{S}^{yt}(u, v) = (\rho(u) \sinh v, h(u), \rho(u) \cosh v), \tag{5.67}$$

and the functions  $\rho(u), h(u)$  have the property of  $h'^2(u) - \rho'^2(u) = -1$ .

Furthermore,

$$\mathcal{S}_u^{yt} = \begin{pmatrix} \rho'(u) \sinh(v) \\ h'(u) \\ \rho'(u) \cosh(v) \end{pmatrix} = \mathbf{n}_u \quad \text{and} \quad \mathcal{S}_v^{yt} = \begin{pmatrix} \rho(u) \cosh(v) \\ 0 \\ \rho(u) \sinh(v) \end{pmatrix} = \rho(u)\mathbf{n}_v, \tag{5.68}$$

resulting in the first fundamental form :

$$I = \begin{pmatrix} -1 & 0 \\ 0 & \rho^2(u) \end{pmatrix}. \quad (5.69)$$

We note that  $\mathcal{S}_u^{yt} = \mathbf{n}_u$  is a unit time-like vector pointing along the meridians, while  $\mathcal{S}_v^{yt} = \rho \mathbf{n}_v$ , such that  $\mathbf{n}_v$  is a unit space-like vector pointing along the parallels. And,  $g(\mathbf{n}_u, \mathbf{n}_v) = 0$ . So we have an orthonormal basis, and hence a unit time-like vector  $t$  tangent to  $\mathcal{S}^{yt}$  can be written  $\mathbf{n}_u \cosh \psi + \mathbf{n}_v \sinh \psi$  where  $\psi$  is the hyperbolic angle between  $t$  and  $\mathbf{n}_u$ .

This time the Lagrangian is

$$- \dot{u}^2 + \rho^2 \dot{v}^2, \quad (5.70)$$

giving Euler-Lagrange equations,

$$\begin{aligned} \ddot{u} &= -\rho \rho' \dot{v}^2 \\ \frac{d}{ds} (\rho^2 \dot{v}) &= 0. \end{aligned} \quad (5.71)$$

Now let  $\gamma$  be a timelike geodesic on  $\mathcal{S}^{yt}$ , given by  $u(s), v(s)$ . Then as before (Sec. (2.4)), we have:

$$\begin{aligned} \dot{\gamma} &= \dot{u} \mathcal{S}_u^{yt} + \dot{v} \mathcal{S}_v^{yt} \\ &= \dot{u} \mathbf{n}_u + \rho \dot{v} \mathbf{n}_v. \end{aligned} \quad (5.72)$$

As before, this gives:

$$\dot{\gamma} = \mathbf{n}_u \cosh \psi + \mathbf{n}_v \sinh \psi \quad (5.73)$$

where  $\psi$  is the hyperbolic angle between  $\dot{\gamma}$  and  $\mathbf{n}_u$ , i.e. between  $\dot{\gamma}$  and a meridian.

We then see that the second Euler-Lagrange equation is equivalent to  $\rho \sinh \psi$  being constant.

Conversely, let  $\gamma$  be a proper-time parametrized curve such that  $\rho \sinh \psi = \rho^2 \dot{v}$  is con-

stant, and  $\dot{u} \neq 0$ . We then have:

$$\dot{u}^2 - \rho^2 \dot{v}^2 = 1 \quad \text{and} \quad \rho^2 \dot{v} = \text{constant}$$

This calculation again is identical to the previous.

It follows that  $\gamma$  is a time-like geodesic.

□

### 5.2.3 Clairaut's Theorem of Surface of Rotation Generated by Null Rotation

The surface of rotation in this case is parametrized by:

$$\mathcal{S}^n(u, v) = \begin{pmatrix} -vq(u) + vh(u) \\ (1 - \frac{v^2}{2})q(u) + \frac{v^2}{2}h(u) \\ -\frac{v^2}{2}q(u) + (1 + \frac{v^2}{2})h(u) \end{pmatrix}, \quad (5.74)$$

where the functions  $q(u), h(u)$  have the property of  $q'^2(u) - h'^2(u) = -1$ . So the curve  $\gamma$  is time-like and parametrized by a proper time. And let  $\rho(u) = q(u) - h(u)$ .

Moreover,

$$\mathcal{S}_u^n = \begin{pmatrix} -vq'(u) + vh'(u) \\ (1 - \frac{v^2}{2})q'(u) + \frac{v^2}{2}h'(u) \\ -\frac{v^2}{2}q'(u) + (1 + \frac{v^2}{2})h'(u) \end{pmatrix} = \mathbf{n}_u, \quad (5.75)$$

also

$$\mathcal{S}_v^n = \begin{pmatrix} -q(u) + h(u) \\ -vq(u) + vh(u) \\ -vq(u) + vh(u) \end{pmatrix} = (q(u) - h(u)) \begin{pmatrix} -1 \\ -v \\ -v \end{pmatrix} = \rho(u)\mathbf{n}_v, \quad (5.76)$$

this is resulting the first fundamental form :

$$I = \begin{pmatrix} -1 & 0 \\ 0 & \rho^2(u) \end{pmatrix}. \quad (5.77)$$

We note that  $\mathcal{S}_u^n = \mathbf{n}_u$  is a unit time-like vector pointing along the meridians, while  $\mathcal{S}_v^n = \rho \mathbf{n}_v$ , such that  $\mathbf{n}_v$  is a unit space-like vector pointing along the parallels. And  $g(\mathbf{n}_u, \mathbf{n}_v) = 0$ . So we have an orthonormal basis, and hence a unit time-like vector  $t$  tangent to  $\mathcal{S}^n$  can be written  $\mathbf{n}_u \cosh \psi + \mathbf{n}_v \sinh \psi$  where  $\psi$  is the hyperbolic angle between  $t$  and  $\mathbf{n}_u$ .

In this case the Lagrangian is

$$-\dot{u}^2 + \rho^2 \dot{v}^2 \quad (5.78)$$

giving the Euler-Lagrange equations,

$$\begin{aligned} \ddot{u} &= -\rho \rho' \dot{v}^2 \\ \frac{d}{ds} (\rho^2 \dot{v}) &= 0. \end{aligned} \quad (5.79)$$

Now let  $\gamma$  be a time-like geodesic on  $\mathcal{S}^n$ , given by  $u(s), v(s)$ . Then we have

$$\begin{aligned} \dot{\gamma} &= \dot{u} \mathcal{S}_u^n + \dot{v} \mathcal{S}_v^n \\ &= \dot{u} \mathbf{n}_u + \rho \dot{v} \mathbf{n}_v. \end{aligned} \quad (5.80)$$

Once more, this gives

$$\dot{\gamma} = \mathbf{n}_u \cosh \psi + \mathbf{n}_v \sinh \psi \quad (5.81)$$

where  $\psi$  is now the hyperbolic angle between  $\dot{\gamma}$  and  $\mathbf{n}_u$ , i.e. between  $\dot{\gamma}$  and a meridian.

We then see that the second Euler-Lagrange equation is equivalent to  $\rho \sinh \psi$  being constant.

Conversely, let  $\gamma$  be a proper-time parametrized curve such that  $\rho \sinh \psi = \rho^2 \dot{v}$  is constant, and  $\dot{u} \neq 0$ . We then have

$$\dot{u}^2 - \rho^2 \dot{v}^2 = 1 \quad \text{and} \quad \rho^2 \dot{v} = \text{constant}$$

This calculation is identical to previous.

It follows that  $\gamma$  is a time-like geodesic.

□

### 5.3 Discussion

In conclusion, we can see that Clairaut's theorem has a Minkowski space analogue with  $\rho \sinh \psi$  replacing  $\rho \sin \psi$  as the quantity conserved along a time-like geodesic. As before, we can immediately deduce that all meridians are geodesics. In addition, we see that for small values of  $\psi$ , since  $\sin \psi \approx \psi \approx \sinh \psi$  the geodesics will be close to those for the Euclidean case.

**Theorem 5.2.** *Let  $\gamma$  be a time-like geodesic curve on a surface of revolution  $\mathcal{S} \in \mathbb{M}^{2,1}$ , and let  $\rho$  be the distance of a point of  $\mathcal{S}$  from the axis of rotation, and let  $\psi$  be the hyperbolic angle between  $\gamma$  and the meridians of  $\mathcal{S}$ . Then  $\rho \sinh \psi$  is constant along  $\gamma$ . Conversely, if  $\rho \sinh \psi$  is constant along some curve  $\gamma$  in the surface, and if no part of  $\gamma$  is part of some parallel of  $\mathcal{S}$ , then  $\gamma$  is a geodesic.*

Since our conserved quantity commutes with  $H$ , the geodesic equations are completely integrable.

In order to find the equations of geodesics explicitly in terms of integrals, we recall the calculation of example (2.13):

**Example 5.3.** *In this case we have the Lagrangian given by:*

$$L = -\dot{u}^2 + \rho^2(u)\dot{v}^2 \quad (5.82)$$

*which is constant along a geodesic, and Euler-Lagrange equation*

$$\frac{d}{ds}(2\rho^2(u)\dot{v}) = 0, \quad (5.83)$$

*so that  $\rho^2\dot{v}$  is constant, say  $\Omega$ . In this case again we have two community conserved quantities, and so the system should be completely integrable.*

*In fact, we can easily find the geodesics, as follows*

$$\dot{v} = \Omega/\rho^2(u) \quad (5.84)$$



and so,

$$L = -\dot{u}^2 + \Omega^2/\rho^2(u) \quad (5.85)$$

giving

$$\dot{u} = \sqrt{\Omega^2/\rho^2(u) - L} \quad (5.86)$$

rearranging of this

$$\frac{du}{\sqrt{\Omega^2/\rho^2(u) - L}} = dt \quad (5.87)$$

so that

$$t = \int \frac{du}{\sqrt{\Omega^2/\rho^2(u) - L}} + C_1. \quad (5.88)$$

This specifies  $u$  as a function of  $t$ .

We now return to  $\rho^2(u)\dot{v} = \Omega$ , and  $u$  is a function of  $t$  obtained above, then

$$\dot{v} = \Omega/\rho^2(u(t)) \quad (5.89)$$

then we conclude

$$v = \int \frac{\Omega}{\rho^2(u(t))} dt + C_2. \quad (5.90)$$

This gives both  $u$  and  $v$  explicitly in terms of integrals.

However, in comparison to the Euclidean case, the characterisation of geodesics in surfaces of revolution looks formally identical in the Euclidean and Minkowskian case. In each case geodesics are completely characterized by  $\rho^2\dot{v}$  being a conserved quantity. In spite of this, the difference in signature results in entirely different qualitative behaviour of the geodesics in these surfaces.

As a consequence, let us now compute an explicit example, to investigate the difference between geodesics in Euclidean and Minkowskian cases. In this case a surface of rotation generated by time-like rotation.

We consider the simplest non-trivial case: the surface of revolution generated by a straight line, given by  $z = 2x$  in the Euclidean, and  $t = 2x$  in the Minkowski case, restricted

to positive values of  $x$ .

In this case, the surface of rotation is actually flat. Nevertheless, the geodesics display distinct quantitative and qualitative behaviour, as we now see.

First, we find the equation of an arc-length parametrized geodesic in the Euclidean case.

An arc-length parametrization of the generator is given by  $x = u/\sqrt{5}$ ,  $z = 2u/\sqrt{5}$ , which gives the metric

$$ds^2 = du^2 + \frac{u^2}{5}dv^2 \quad (5.91)$$

and so the Lagrangian

$$L = \dot{u}^2 + \frac{u^2}{5}\dot{v}^2. \quad (5.92)$$

A geodesic is then completely determined by the value of  $u^2\dot{v} = \Omega$ , and the condition that  $L = 1$ . Substituting for  $\Omega$  in  $L$  gives

$$\dot{u}^2 = 1 - \frac{\Omega^2}{5u^2} \quad (5.93)$$

and hence

$$\frac{\dot{u}^2}{\dot{v}^2} = \frac{u^2}{\Omega^2} \left( u^2 - \frac{\Omega^2}{5} \right) \quad (5.94)$$

so that the differential equation

$$\frac{du}{dv} = \pm \frac{u}{\Omega} \sqrt{u^2 - \Omega^2/5} \quad (5.95)$$

describes the curve in the  $(v, u)$  plane which gives a geodesic in the surface of rotation.

In the Minkowskian case, we have an arc-length parametrization of the generator given by  $x = u/\sqrt{3}$ ,  $t = 2u/\sqrt{3}$ , which gives the metric

$$ds^2 = -du^2 + \frac{u^2}{3}dv^2 \quad (5.96)$$

and

$$L = -\dot{u}^2 + \frac{u^2}{3}\dot{v}^2. \quad (5.97)$$

The analogous calculation with  $L = -1$ , then gives:

$$\frac{du}{dv} = \pm \frac{u}{\Omega} \sqrt{u^2 + \Omega^2/3}. \quad (5.98)$$

We are interested in time-like geodesics, for which  $ds^2 < 0$ ; thus we must have

$$du^2 > \frac{u^2}{3} dv^2 \quad (5.99)$$

or, equivalently,

$$\left(\frac{du}{dv}\right)^2 > \frac{u^2}{3}. \quad (5.100)$$

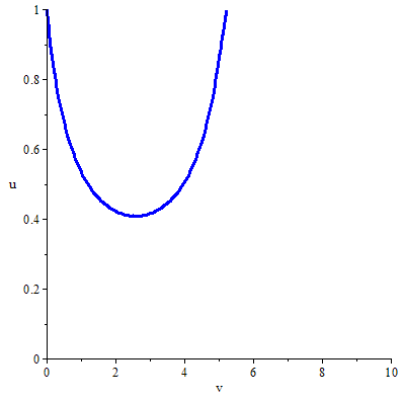
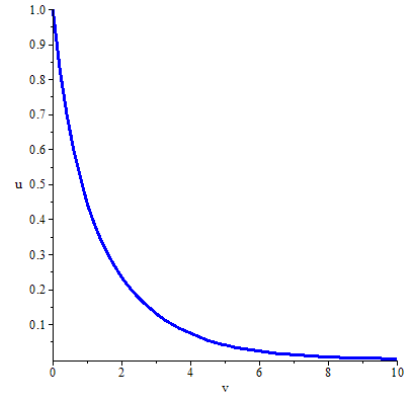
We can therefore ensure that a geodesic is time-like by insisting that its initial value of  $du/dv$  satisfy this criterion.

An immediate qualitative difference is that in the Euclidean case, all geodesics except the generators can be continued for arbitrarily large positive or negative values of the parameter, have a closest point of approach to the origin determined by  $\Omega$ , and are symmetric about this point. This is a consequence of the fact that since  $\rho \sin \theta$  is constant, and  $\sin \theta$  is bounded above by 1, there is a minimum possible value of  $\rho$ . In the Minkowskian case, we have  $\dot{u} > 0$ , so a time-like geodesic cannot bounce away from the origin: but since there is no upper bound on  $\sinh \theta$ , the radial distance  $\rho$  can become arbitrarily small. Hence all time-like geodesics approach the origin arbitrarily.

To illustrate this behaviour, we consider geodesics starting at  $v = 0, u = 1$  with initial gradient  $du/dv = -1$ , so that the Minkowskian geodesic is time-like. In the Euclidean case, we obtain  $\Omega = \sqrt{5/6}$ , and in the Minkowski case,  $\Omega = \sqrt{3/2}$ . The geodesics cannot be obtained in an instructive closed form, but can be found numerically. The results are shown in Figures (5.1) and (5.2).

We see in Figure (5.1) how the downward geodesic in the Euclidean case has a minimum value of  $u$  at  $u = 1/\sqrt{6}$ ; after this it proceeds back up, with the sign changed in the differential equation.

By contrast, Figure (5.2) shows that in the Minkowskian case, the geodesic passes down

Fig. 5.1: Geodesic in  $\mathbb{E}^3$ Fig. 5.2: Geodesic in  $\mathbb{M}^{2,1}$ 

arbitrarily close to the origin.

It is clear that this difference is generic. In any surface of revolution (other than the cylinder) there will be geodesics in the Euclidean case which bounce away from regions of sufficiently small  $\rho$ ; on the other hand, in the Minkowskian case, since  $\dot{u}^2 - \rho(u)^2 \dot{v}^2 = 1$ , it follows that  $|\dot{u}| > 1$ , so no such bouncing can take place, and timelike geodesics will generally reach every value of  $u$  in the domain of the generating curve.



## 6. SURFACES OF ROTATION AND THEIR GENERALIZATION OF 4D MINKOWSKI SPACE

This chapter provides a brief description of surfaces of rotation of four-dimensional Minkowski space, defined above in (3.1).

Firstly, we need to produce the matrices of rotation corresponding to the appropriate subgroup of the Lorentz group, and then generate surfaces of rotation.

### 6.1 Introduction

As in  $\mathbb{E}^3$ , the matrices of rotations in  $\mathbb{E}^4$  preserve all distances and all inner products are preserved. The analogue of a matrix of rotation in  $\mathbb{M}^{3,1}$  with standard basis  $e_x, e_y, e_z, e_t$ , is denoted by  $\mathcal{M}$ .

The rotation matrices are replaced by Lorentz transformation such that:

$$\mathcal{M}^T \eta \mathcal{M} = \eta, \quad (6.1)$$

where,  $\mathcal{M}^T$  is the transpose, and  $\eta$  is the metric matrix of 4D Minkowski space given by :

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (6.2)$$

The set of all  $4 \times 4$  matrices which satisfies the property above (6.1) is denoted by  $O(3, 1)$ . If, in addition,  $\det(\mathcal{M}) = 1$  and  $\mathcal{M}^{4,4} \leq -1$ , we have the group of proper orthochronous Lorentz transformations, denoted here by  $SO(3, 1)$ .

In this section we will provide different types of matrices of rotations, which are the subgroups of  $SO(3, 1)$  corresponding to rotation about a chosen axis in  $E^4$ . In particular, we will choose the two parameter matrices groups of rotations, since a two parameter group acts on a curve to give a hypersurfaces of dimension 3.

The Lorentz group is a subgroup of the diffeomorphism group of  $\mathbb{M}^{3,1}$ , and its Lie algebra can be identified with vector fields on  $\mathbb{M}^{3,1}$ . In particular, Killing vector fields are the vectors which generate the isometries on space. We can immediately (as in chapter 4 (4.3)) write down the general vector fields :

$$V := \xi(x, y, z, t) \frac{\partial}{\partial x} + \eta(x, y, z, t) \frac{\partial}{\partial y} + \zeta(x, y, z, t) \frac{\partial}{\partial z} + \tau(x, y, z, t) \frac{\partial}{\partial t}, \quad (6.3)$$

where  $\xi(x, y, z, t), \eta(x, y, z, t), \zeta(x, y, z, t)$  and  $\tau(x, y, z, t)$  are real functions.

We are seeking these functions  $\xi(x, y, z, t), \eta(x, y, z, t), \zeta(x, y, z, t)$  and  $\tau(x, y, z, t)$ , such that, the vector field (6.3) satisfies the Killing vector field equation.

$$g_{ac} V_{;b}^c + g_{cb} V_{;a}^c = 0. \quad (6.4)$$

Similarly as in chapter 4 equation (4.19), the expression of the Killing equation in terms of these components is:

$$\xi_x = \eta_y = \zeta_z = \tau_t = 0 \quad (6.5a)$$

$$\xi_y + \eta_x = \xi_t - \tau_x = \eta_t - \tau_y = \zeta_x + \xi_z = \zeta_y + \eta_z = \zeta_t - \tau_z = 0. \quad (6.5b)$$

If we search for the function  $\eta$ , then from (6.5) we have:

$$\xi_y + \eta_x = 0 \quad (6.6)$$

then differentiating with respect to  $x$  we get:

$$\xi_{yx} + \eta_{xx} = 0, \quad (6.7)$$

which gives  $\eta_{xx} = 0$ .

Therefore the function  $\eta$  can be given by:

$$\eta(x, z, t) = f(z, t)x + g(z, t), \quad (6.8)$$

where,  $f(z, t)$  and  $g(z, t)$  are functions of  $z, t$ .

Similarly, differentiation with respect to  $t$  of

$$\eta_t - \tau_y = 0 \quad (6.9)$$

gives  $\eta_{tt} = 0$ .

And from (6.8) we have,

$$\eta_{tt} = f_{tt}(z, t)x + g_{tt}(z, t) = 0, \quad (6.10)$$

this means  $f_{tt}(z, t) = g_{tt}(z, t) = 0$ , or

$$f(z, t) = h_1(z)t + i_1(z) \text{ and } g(z, t) = h_2(z)t + i_2(z) \quad (6.11)$$

Substituting this equation into (6.8), we obtain:

$$\eta(x, z, t) = (h_1(z)t + i_1(z))x + h_2(z)t + i_2(z) \quad (6.12)$$

again, differentiation with respect to  $z$  of:

$$\zeta_y + \eta_z = 0 \quad (6.13)$$



gives  $\eta_{zz} = 0$ .

And from (6.12) , we have:

$$\eta_{zz} = (h_{1zz}(z)t + i_{1zz}(z))x + h_{2zz}(z)t + i_{2zz}(z) = 0, \quad (6.14)$$

This means  $h_{1zz}(z) = h_{2zz}(z) = i_{1zz}(z) = i_{2zz}(z) = 0$ , or  $h_{1z}(z), h_{2z}(z), i_{1z}(z)$  and  $i_{2z}(z)$  all are constants.

So, they can be given by:

$$h_1(z) = a_1z + b_1, \quad i_1(z) = a_2z + b_2, \quad h_2(z) = a_3z + b_3 \quad \text{and} \quad i_2(z) = a_4z + b_4, \quad (6.15)$$

where  $a_1, a_2, a_3, a_4, b_1, b_2, b_3$  and  $b_4$  are arbitrary constants.

Now substituting (6.15) into (6.12) we have the function  $\eta$  given by :

$$\eta(x, z, t) = ((a_1z + b_1)t + (a_2z + b_2))x + (a_3z + b_3)t + a_4z + b_4 \quad (6.16)$$

In the same way from (6.5), if we search  $\xi$  then take the equations

$$\xi_y + \eta_x = 0, \quad \xi_t - \tau_x = 0 \quad \text{and} \quad \zeta_x + \xi_z = 0, \quad (6.17)$$

and differentiation with respect to  $y, t$  and  $z$  respectively gives:

$$\xi_{yy} = \xi_{tt} = \xi_{zz} = 0, \quad (6.18)$$

then, with the same calculation as before, we obtain:

$$\xi(y, z, t) = ((c_1z + d_1)t + (c_2z + d_2))y + (c_3z + d_3)t + c_4z + d_4, \quad (6.19)$$

where  $c_1, c_2, c_3, c_4, d_1, d_2, d_3$  and  $d_4$  are arbitrary constants.

Also, if we seek  $\zeta$  then from the equations

$$\zeta_x + \xi_z = 0 \quad , \quad \zeta_y + \eta_z = 0 \quad \text{and} \quad \zeta_t - \tau_z = 0, \quad (6.20)$$

and differentiation with respect to  $x, y$  and  $t$  respectively we get:

$$\zeta_{xx} = \zeta_{yy} = \zeta_{tt} = 0. \quad (6.21)$$

Then, the function  $\zeta(x, y, t)$  is given by:

$$\zeta(x, y, t) = ((a_5y + b_5)t + (a_6y + b_6))x + (a_7y + b_7)t + a_8y + b_8, \quad (6.22)$$

where  $a_5, a_6, a_7, a_8, b_5, b_6, b_7$  and  $b_8$  are arbitrary constants.

Finally, the function  $\tau(x, y, z)$  can be obtained from the equations

$$\xi_t - \tau_x = 0 \quad , \quad \eta_t - \tau_y = 0 \quad \text{and} \quad \zeta_t - \tau_z = 0, \quad (6.23)$$

and differentiation with respect to  $x, y$  and  $z$  respectively gives:

$$\tau_{xx} = \tau_{yy} = \tau_{zz} = 0. \quad (6.24)$$

Then, by the same calculation, the function  $\tau$  is given by:

$$\tau(y, z, t) = ((c_5y + d_5)z + (c_6y + d_6))x + (c_7y + d_7)z + c_8y + d_8, \quad (6.25)$$

where  $c_5, c_6, c_7, c_8, d_5, d_6, d_7$  and  $d_8$  are arbitrary constants.

From (6.16) , (6.19), (6.22) and (6.25), the functions  $\xi, \eta, \zeta$  and  $\tau$  can be explicitly given

by :

$$\begin{aligned}
\xi(y, z, t) &= c_1 zty + d_1 ty + c_2 zy + d_2 y + c_3 zt + d_3 t + c_4 z + d_4, \\
\eta(x, z, t) &= a_1 ztx + b_1 tx + a_2 zx + b_2 x + a_3 zt + b_3 t + a_4 z + b_4, \\
\zeta(x, y, t) &= a_5 ytx + b_5 tx + a_6 yx + b_6 x + a_7 yt + b_7 t + a_8 y + b_8, \\
\tau(y, z, t) &= c_5 yzx + d_5 zx + c_6 yx + d_6 x + c_7 yz + d_7 z + c_8 y + d_8.
\end{aligned} \tag{6.26}$$

Now, substituting (6.26) into (6.5b), we obtain;

$$\begin{aligned}
c_1 &= a_1 = a_5 = c_5 = d_1 = b_1 = b_5 = d_5 = c_2 \\
&= a_2 = a_6 = c_6 = c_3 = a_3 = a_7 = c_7 = 0, \\
&\text{and} \\
d_2 &= -b_2, \quad d_3 = d_6, \quad b_3 = c_8, \\
b_6 &= -c_4, \quad a_8 = -a_4, \quad b_7 = d_7.
\end{aligned} \tag{6.27}$$

Then, substituting (6.27) into (6.26) we can deduce that the functions  $\xi, \eta, \zeta$  and  $\tau$  can be given by :

$$\begin{aligned}
\xi &= -b_2 y + c_4 z + d_3 t + d_4 \\
\eta &= b_2 x - a_8 z + b_3 t + b_4 \\
\zeta &= -c_4 x + a_8 y + b_6 t + b_8 \\
\tau &= d_3 x + b_3 y + b_6 z + d_8.
\end{aligned} \tag{6.28}$$

Now, substituting (6.28) into (6.3), we get the general solution of the Killing vector field equation, which gives the full symmetry of special relativity, includes translations, rotations and boosts. However, we are interested in those transformation which fix some "axis of rotation" passing through the origin. Therefore the part of translations will be omitted. That means  $d_4 = b_4 = b_8 = d_8 = 0$ .

So, the functions  $\xi, \eta, \zeta$  and  $\tau$  can be given in final form by :

$$\begin{aligned}\xi &= -b_2y + c_4z + d_3t \\ \eta &= b_2x - a_8z + b_3t \\ \zeta &= -c_4x + a_8y + b_6t \\ \tau &= d_3x + b_3y + b_6z.\end{aligned}\tag{6.29}$$

Now, let  $b_2 = \alpha, a_8 = \beta, c_4 = \gamma, d_3 = \delta, b_3 = \epsilon$  and  $b_6 = \varepsilon$ , then substituting (6.29) into (6.3) we have the general Killing vector fields given by :

$$\begin{aligned}V &= \alpha(-y\partial_x + x\partial_y) + \beta(-z\partial_y + y\partial_z) + \gamma(-x\partial_z + z\partial_x) \\ &\quad + \delta(x\partial_t + t\partial_x) + \epsilon(y\partial_t + t\partial_y) + \varepsilon(z\partial_t + t\partial_z),\end{aligned}\tag{6.30}$$

where  $\alpha, \beta, \gamma, \delta, \epsilon$  and  $\varepsilon$  are constants.

Now, we obtain a basis for the space of Killing vector fields:

- Vector fields on  $\mathbb{M}^{3,1}$  generating three rotations

$$R_z = -y\partial_x + x\partial_y, \quad R_x = -z\partial_y + y\partial_z, \quad R_y = -x\partial_z + z\partial_x,$$

where,  $R_x, R_y$  and  $R_z$  are the rotations fixing  $t$  around  $x, y$  and  $z$  axes respectively.

- Vector fields in  $\mathbb{M}^{3,1}$  generating three boosts

$$B_x = x\partial_t + t\partial_x, \quad B_y = y\partial_t + t\partial_y, \quad B_z = z\partial_t + t\partial_z,$$

where  $B_x, B_y$  and  $B_z$  are the boosts in direction of  $x, y$  and  $z$  axes respectively.

These six give the rotations and boosts which form a basis for the Lie algebra of Killing vector fields.

It is also useful to consider the generators of the null rotations.

- Vector fields in  $\mathbb{M}^{3,1}$  generating null rotation, here we only consider the two generators that have an axis of rotation located in  $zt$ - plane . So the infinitesimal generators of these null rotations are:

$$N_x = B_x - R_y = x(\partial_t + \partial_z) + (t - z)\partial_x \quad , \quad N_y = B_y + R_x = y(\partial_t + \partial_z) + (t - z)\partial_y,$$

where  $N_x$  and  $N_y$  are the null rotations around the  $z = t$  axis, with axes of rotation  $x = 0, t = z$  or  $y = 0, t = z$

It maybe helpful if we recall how to obtain a one parameter group from a base of vector field. For instance, if we write down the infinitesimal generator given by the rotation :

$$R_z = -y\partial_x + x\partial_y \quad (6.31)$$

As in (4.4.1) the Killing vector field is given by

$$V^a = \begin{pmatrix} -y & x & 0 & 0 \end{pmatrix}^T. \quad (6.32)$$

So, the  $4 \times 4$  matrix  $M$ ; corresponding to the infinitesimal generator, can be given in  $(x, y, z, t)$  coordinates by :

$$M = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (6.33)$$

Therefore, the one parameter subgroup of rotation matrices (in this case) is :

$$\mathcal{M}(\beta) = e^{\beta M} = \begin{pmatrix} \cos(\beta) & -\sin(\beta) & 0 & 0 \\ \sin(\beta) & \cos(\beta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (6.34)$$

This matrix fixes the "plane" of rotation  $zt$ .

So, we can recognise the one parameter group of rotations of the other generators. But we will make use of the following generators to obtain two parameter groups:

- Two parabolic (Null rotations in  $zt$ - plane.),
- Three hyperbolic; ( we only consider the boost of  $B_z = z\partial_t + t\partial_z$  ),
- Three elliptic(rotation about  $x, y, z$  axes respectively). In this case we will consider only the rotation around  $z$  axis, which is generated by  $R_z = -y\partial_x + x\partial_y$  .

Now, we will provide these relevant cases of the one parameter subgroup of  $SO(3, 1)$  representing Lorentz transformations.

### 1. Parabolic

In this case there are two generators of interest. The first generator is

$$N_x = x(\partial_t + \partial_z) + (t - z)\partial_x \quad (6.35)$$

which gives the one parameter matrix group of rotations by:

$$\mathcal{M}_1(\alpha) = e^{\alpha M_1} = \begin{pmatrix} 1 & 0 & -\alpha & \alpha \\ 0 & 1 & 0 & 0 \\ \alpha & 0 & 1 - \frac{\alpha^2}{2} & \frac{\alpha^2}{2} \\ \alpha & 0 & -\frac{\alpha^2}{2} & 1 + \frac{\alpha^2}{2} \end{pmatrix} \quad (6.36)$$

The second generator is

$$N_y = y(\partial_t + \partial_z) + (t - z)\partial_y \quad (6.37)$$

which gives a one parameter matrix group of rotations by :

$$\mathcal{M}_2(\alpha) = e^{\alpha M_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\alpha & \alpha \\ 0 & \alpha & 1 - \frac{\alpha^2}{2} & \frac{\alpha^2}{2} \\ 0 & \alpha & -\frac{\alpha^2}{2} & 1 + \frac{\alpha^2}{2} \end{pmatrix}. \quad (6.38)$$

## 2. Hyperbolic

The vector field on  $\mathbb{M}^{3,1}$  is given by the generator:

$$B_z = z\partial_t + t\partial_z \quad (6.39)$$

which gives a one parameter matrix group of rotations by :

$$\mathcal{M}_3(\beta) = e^{\beta M_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh(\beta) & \sinh(\beta) \\ 0 & 0 & \sinh(\beta) & \cosh(\beta) \end{pmatrix}. \quad (6.40)$$

## 3. Elliptic

We listed three generators above. The first generator is

$$R_z = -y\partial_x + x\partial_y \quad (6.41)$$

which gives a one parameter matrix group of rotations by :

$$\mathcal{M}_4(\beta) = e^{\beta M_4} = \begin{pmatrix} \cos(\beta) & -\sin(\beta) & 0 & 0 \\ \sin(\beta) & \cos(\beta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (6.42)$$

The other two matrices of elliptic are given by the generators

$$R_y = -x\partial_z + z\partial_x \quad , \quad R_x = -z\partial_y + y\partial_z$$

They are similar to the above, but we are not considering them in this work.

## 6.2 Generating Two Parameter Subgroup of $SO(3, 1)$ Which are Analogues of Rotations in $\mathbb{E}^3$

The sub-algebra of the Lie algebra of the Lorentz group can be enumerated, up to conjugacy, from which we can list the closed subgroup of the Lorentz group, up to conjugacy, see [40] chapter six for sub-algebras of the Lie algebra of the Lorentz group.

We are seeking two parameter subgroups of  $SO(3, 1)$  which are analogues of one parameter group of rotations in  $\mathbb{E}^3$ . So, we are going to find two parameter subgroup which fix some axis of rotation. Then we find two dimensional sub-algebras, and hence the corresponding subgroups. Therefore, we have three cases:

**Case 1** Two parameter group fixing the null axis located in  $zt$ - plane given by  $(0, 0, 1, 1)$ .

Substitute this into the Killing vector fields equation (6.30); we deduce that:

$$\gamma = -\delta, \beta = \epsilon \quad \text{and} \quad \varepsilon = 0 \quad (6.43)$$

So the Killing Vector field which is vanishing on  $(0, 0, 1, 1)$  is given by :

$$V = \alpha(-y\partial_x + x\partial_y) + \beta(-z\partial_y + y\partial_z) + \gamma(-x\partial_z + z\partial_x) - \gamma(x\partial_t + t\partial_x) + \beta(y\partial_t + t\partial_y). \quad (6.44)$$

Or,

$$V = \alpha(-y\partial_x + x\partial_y) + \beta(-z\partial_y + y\partial_z + y\partial_t + t\partial_y) + \gamma(-x\partial_z + z\partial_x - x\partial_t - t\partial_x). \quad (6.45)$$



So,

$$V = \alpha R_z + \gamma N_x + \beta N_y, \quad (6.46)$$

which is a three dimensional sub-space.

But, we are interested in finding a two dimensional sub-algebra to give a two parameter subgroup. Therefore; we need to find two vectors which give a closed sub-algebra.

Thus,

$$\begin{aligned} [N_x, N_y] &= [B_x - R_y, B_y + R_x] = [B_x, B_y + R_x] - [R_y, B_y + R_x] \\ &= [B_x, B_y] + [B_x, R_x] - [R_y, B_y] - [R_y, R_x] \\ &= R_z + 0 - 0 - R_z = 0, \end{aligned} \quad (6.47)$$

$\therefore \{N_x, N_y\}$  is a closed sub-algebra and it is also Abelian.

Also,

$$[R_z, N_y] = [R_z, B_y + R_x] = [R_z, B_y] + [R_z, R_x] = B_x + R_y. \quad (6.48)$$

This is not in  $Sp\{R_z, N_y\}$ .

$\therefore$  this is not a closed sub-algebra.

Also,

$$[R_z, N_x] = [R_z, B_x - R_y] = [R_z, B_x] - [R_z, R_y] = B_y - R_x. \quad (6.49)$$

This is not in  $Sp\{R_z, N_x\}$ .

$\therefore$  this is not closed sub-algebra.

So, we choose  $\{N_x, N_y\}$  as a basis . Thus we have an abelian subgroup of  $SO(3, 1)$ .

Then  $N_x, N_y$  generate an abelian sub-algebra consisting entirely of parabolic. So the matrices  $\mathcal{M}_1, \mathcal{M}_2$  will make the rotational group of matrices for this case.

**Case 2** The two parameter group fixing a space-like axis say the line given by  $(0, 1, 0, 0)$  i.e. the  $y$ -axis.

Substituting this into the Killing vector field equation (6.30), we have:

$$\alpha = \beta = \epsilon = 0. \quad (6.50)$$

So, the Killing vector field becomes :

$$V = \gamma(-x\partial_z + z\partial_x) + \delta(x\partial_t + t\partial_x) + \varepsilon(z\partial_t + t\partial_z). \quad (6.51)$$

Or,

$$V = \gamma R_y + \delta B_x + \varepsilon B_z. \quad (6.52)$$

again, they form a three dimensional sub-space. Closed under Lie brackets, and so a sub-algebra. But, we need two a dimensional closed sub-algebra. but

$$\begin{aligned} [R_y, B_x] &= z\partial_t + t\partial_z = B_z && \text{is not closed sub-algebra.} \\ [B_x, B_z] &= -x\partial_z + z\partial_x = R_y && \text{is not closed sub-algebra.} \\ [R_y, B_z] &= -x\partial_t - t\partial_x = -B_x && \text{is not closed sub-algebra.} \end{aligned} \quad (6.53)$$

So, there is no two dimensional sub-algebra; with a basis consisting of a subset  $\{R_y, B_x, B_z\}$

But lets recall  $N_x = B_x - R_y$ , also consider  $\tilde{N}_x = B_x + R_y$ . Then, the equation (6.52) equivalent to:

$$V = \gamma N_x + \delta \tilde{N}_x + \varepsilon B_z. \quad (6.54)$$

Since,

$$\begin{aligned} [N_x, \tilde{N}_x] &= [B_x - R_y, B_x + R_y] = [B_x, B_x + R_y] - [R_y, B_x + R_y] \\ &= [B_x, B_x] + [B_x, R_y] - [R_y, B_x] - [R_y, R_y] = B_z. \end{aligned} \quad (6.55)$$

And,

$$\begin{aligned} [\tilde{N}_x, B_z] &= [B_x + R_y, B_z] = [B_x, B_z] + [R_y, B_z] \\ &= -R_y - B_x = -\tilde{N}_x. \end{aligned} \quad (6.56)$$

Also,

$$\begin{aligned} [N_x, B_z] &= [B_x - R_y, B_z] = [B_x, B_z] - [R_y, B_z] \\ &= -R_y + B_x = N_x. \end{aligned} \quad (6.57)$$

We see that  $\{\tilde{N}_x, B_z\}$  and  $\{N_x, B_z\}$  each span a two dimensional sub-algebra.

So, we choose  $\{N_x, B_z\}$  as a basis. And we have (an nonabelian) subgroup of  $SO(3, 1)$ .

Then  $N_x, B_z$  generate a non-abelian sub-algebra isomorphic to the Lie algebra of the affine group  $A(1)$  [41]. In this case the matrices of rotation are given by  $\mathcal{M}_1 \mathcal{M}_3$ , or  $\mathcal{M}_3 \mathcal{M}_1$ . Since this case is not a commutative group.

The two products are different. However, given  $\mathcal{M}_1 \mathcal{M}_3$  there exist  $\mathcal{M}'_1$  and  $\mathcal{M}'_3$  such that  $\mathcal{M}_1 \mathcal{M}_3 = \mathcal{M}'_3 \mathcal{M}'_1$

**Case 3** The two parameter group fixing a time-like axis is given by  $(0, 0, 0, 1)$ .

Substituting the axis of rotation  $l = (0, 0, 0, 1)$  in the Killing vector fields equation (6.30), we get:

$$\delta = \epsilon = \varepsilon = 0. \quad (6.58)$$

So, the Killing vector field becomes:

$$V = \alpha(-y\partial_x + x\partial_y) + \beta(-z\partial_y + y\partial_z) + \gamma(-x\partial_z + z\partial_x), \quad (6.59)$$

Or,

$$V = \alpha R_z + \beta R_x + \gamma R_y, \quad (6.60)$$

again, these constitute a three dimensional sub-space.

While we need a two dimensional sub-algebra.

As previously,

$$\begin{aligned}
 [R_z, R_x] &= R_y && \text{is not a closed sub-algebra.} \\
 [R_x, R_y] &= R_z && \text{is not a closed sub-algebra.} \\
 [R_y, R_z] &= R_x && \text{is not a closed sub-algebra.}
 \end{aligned} \tag{6.61}$$

But this time there is no two dimensional sub-algebra, see [[42],P87].

However,  $SO(3)$  is acting on a point in each surface of constant  $t$  gives a two dimensional sphere, so surface of rotation about  $t$ -axis.

**Case 4**  $R_x, R_y$  and  $R_z$  are a three dimensional sub-algebra, they generate the group  $SO(3)$ , and  $SO(3)$  acting on a point gives a two-dimensional surfaces.

In fact, the surface of rotation is parametrized by fixing  $t(w)$  axis and sphere of radius  $z(w)$  in the plane  $(x, y, z, t(w))$ . This is the spherical symmetric case.

**Case 5** There is another two dimensional sub-algebra known by classification [ see Hall's book [40],P163] in table 6.1 there are three groups of two dimensional sub-algebras. The first and second groups of two dimensional sub-algebras are equivalent to case(1) and case(2) above. And the third one is generated by boost and rotation which is here given by  $R_z, B_z$ .

Therefore:

$$[R_z, B_z] = 0 \tag{6.62}$$

So, we choose  $\{R_z, B_z\}$  as a basis . And we have an abelian subgroup of  $SO(3, 1)$ .

Then  $R_z, B_z$  generate an abelian sub-algebra consisting of boost and rotation. So the matrices  $\mathcal{M}_3, \mathcal{M}_4$  will make the rotational group of matrices for this case.

Note that, this subgroup does not fix any axis, and so it is not a rotation about any axis. But we can still investigate the geodesic on an invariant surface.

In fact, we have for this case a combination of two surfaces of rotation from  $\mathbb{M}^{2,1}$ ; rotation around  $t$  axis in  $(x, y, t)$  and boost in  $z$  direction in  $(z, t)$  plane.

Now we will use these three cases above to generate special surfaces of rotations in Minkowski space.

**Definition 6.1.** *The surface  $\Sigma$  in  $\mathbb{M}^{3,1}$  is called a surface of rotation if  $\Sigma$  is invariant by one of the five cases of two dimensional sub-groups above.*

### 6.3 Surface of Rotation Generated by Two Parabolic Subgroups

The surface of rotation in this case is generated by two parabolic subgroups. We assume that the null axis is located in the  $zt$ - plane, so the axis of rotation is given by  $l = (0, 0, 1, 1)$ , as in case one above. So the matrices of rotations of this surface are  $\mathcal{M}_1$  and  $\mathcal{M}_2$ . We are interested in taking a planar curve  $\gamma$  and rotating it with the corresponding two dimensional sub-groups.

So, for any choice of any arbitrary point  $\{x, y, z, t\}$  we consider the orbits of this point under the group elements  $\mathcal{M}_1(u).\mathcal{M}_2(v)$  to be:

$$\mathcal{M}_1(u).\mathcal{M}_2(v) \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}, \quad (6.63)$$

or

$$= \begin{pmatrix} 1 & 0 & -u & u \\ 0 & 1 & 0 & 0 \\ u & 0 & 1 - \frac{u^2}{2} & \frac{u^2}{2} \\ u & 0 & -\frac{u^2}{2} & 1 + \frac{u^2}{2} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -v & v \\ 0 & v & 1 - \frac{v^2}{2} & \frac{v^2}{2} \\ 0 & v & -\frac{v^2}{2} & 1 + \frac{v^2}{2} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}, \quad (6.64)$$

gives:

$$\begin{pmatrix} x - zu + ut \\ y - vz + vt \\ ux + vy + z - 1/2 zv^2 - 1/2 zu^2 + 1/2 tv^2 + 1/2 tu^2 \\ ux + vy - 1/2 zu^2 - 1/2 zv^2 + t + 1/2 tv^2 + 1/2 tu^2 \end{pmatrix}. \quad (6.65)$$

If  $z = t$ , then the point is fixed, so the surface will not be regular. We therefore assume that  $t \neq z$  then, there exist  $u, v$  such that:

$$\mathcal{M}_1(u) \cdot \mathcal{M}_2(v) \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \tilde{z} \\ \tilde{t} \end{pmatrix}, \quad (6.66)$$

Therefore, with out loss of generality, we take the planar curve  $\gamma$  for this surface of rotation  $\Sigma^1(w, u, v)$  to be the intersection of  $\Sigma^1(w, u, v)$  with  $x = y = 0$ . Then we assume that the curve  $\gamma$  lies in the  $zt$ - plane. Hence, it can be parametrized by:

$$\gamma(w) = (0, 0, z(w), t(w)), \quad (6.67)$$

where  $z(w), t(w)$  are smooth functions. To ensure that the surface is regular, we require that,  $t(w) - z(w)$  is a positive function. Hence, the surface of rotation which will be denoted in this case by  $\Sigma^1$ , around the line  $z = t, x = y = 0$  it can be parametrized by:

$$\Sigma^1(w, u, v) = \mathcal{M}_1(u) \cdot \mathcal{M}_2(v) \cdot \gamma(w), \quad (6.68)$$

or

$$\Sigma^1(w, u, v) = \begin{pmatrix} 1 & 0 & -u & u \\ 0 & 1 & 0 & 0 \\ u & 0 & 1 - \frac{u^2}{2} & \frac{u^2}{2} \\ u & 0 & -\frac{u^2}{2} & 1 + \frac{u^2}{2} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -v & v \\ 0 & v & 1 - \frac{v^2}{2} & \frac{v^2}{2} \\ 0 & v & -\frac{v^2}{2} & 1 + \frac{v^2}{2} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ z(w) \\ t(w) \end{pmatrix}, \quad (6.69)$$

or,

$$\Sigma^1(w, u, v) = \begin{pmatrix} 1 & 0 & -u & u \\ 0 & 1 & -v & v \\ u & v & 1 - 1/2 v^2 - 1/2 u^2 & 1/2 v^2 + 1/2 u^2 \\ u & v & -1/2 u^2 - 1/2 v^2 & 1 + 1/2 v^2 + 1/2 u^2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ z(w) \\ t(w) \end{pmatrix}. \quad (6.70)$$

So, the surface of rotation of this case is :

$$\Sigma^1(w, u, v) = \begin{pmatrix} -uz(w) + ut(w) \\ -vz(w) + vt(w) \\ \left(1 - \frac{v^2}{2} - \frac{u^2}{2}\right) z(w) + \left(\frac{v^2}{2} + \frac{u^2}{2}\right) t(w) \\ \left(-\frac{v^2}{2} - \frac{u^2}{2}\right) z(w) + \left(1 + \frac{v^2}{2} + \frac{u^2}{2}\right) t(w) \end{pmatrix}. \quad (6.71)$$

Also,

$$\Sigma_w^1 = \begin{pmatrix} -uz'(w) + ut'(w) \\ -vz'(w) + vt'(w) \\ \left(1 - \frac{v^2}{2} - \frac{u^2}{2}\right) z'(w) + \left(\frac{v^2}{2} + \frac{u^2}{2}\right) t'(w) \\ \left(-\frac{v^2}{2} - \frac{u^2}{2}\right) z'(w) + \left(1 + \frac{v^2}{2} + \frac{u^2}{2}\right) t'(w) \end{pmatrix}, \quad (6.72)$$

and,

$$\Sigma_u^1 = \begin{pmatrix} -z(w) + t(w) \\ 0 \\ -uz(w) + ut(w) \\ -uz(w) + ut(w) \end{pmatrix}, \quad (6.73)$$

also,

$$\Sigma_v^1 = \begin{pmatrix} 0 \\ -z(w) + t(w) \\ -vz(w) + vt(w) \\ -vz(w) + vt(w) \end{pmatrix}. \quad (6.74)$$

We need to compute the first fundamental form of a surface  $\Sigma$  generated by the three parameters  $(w, u, v)$ ; which is defined by:

$$I_\Sigma = \begin{pmatrix} g(\Sigma_w, \Sigma_w) & g(\Sigma_w, \Sigma_u) & g(\Sigma_w, \Sigma_v) \\ g(\Sigma_u, \Sigma_w) & g(\Sigma_u, \Sigma_u) & g(\Sigma_u, \Sigma_v) \\ g(\Sigma_v, \Sigma_w) & g(\Sigma_v, \Sigma_u) & g(\Sigma_v, \Sigma_v) \end{pmatrix}. \quad (6.75)$$

So from the surface  $\Sigma^1(w, u, v)$  above, we have:

$$\begin{cases} g(\Sigma_w^1, \Sigma_w^1) = z'^2(w) - t'^2(w) \\ g(\Sigma_w^1, \Sigma_u^1) = 0 \\ g(\Sigma_w^1, \Sigma_v^1) = 0 \\ g(\Sigma_u^1, \Sigma_u^1) = (-z(w) + t(w))^2 \\ g(\Sigma_u^1, \Sigma_v^1) = 0 \\ g(\Sigma_v^1, \Sigma_v^1) = (-z(w) + t(w))^2. \end{cases} \quad (6.76)$$



Therefore, the first fundamental form of this surface of rotation is :

$$I_{\Sigma^1} = \begin{pmatrix} z'^2(w) - t'^2(w) & 0 & 0 \\ 0 & (-z(w) + t(w))^2 & 0 \\ 0 & 0 & (-z(w) + t(w))^2 \end{pmatrix}. \quad (6.77)$$

If  $\gamma$  is space-like, we get Riemannian metric. But we are interested in a Lorentzian metric. So we take  $\gamma$  to be time-like. Therefore, we can assume that  $z'^2(w) - t'^2(w) = -1$ . Also define  $\rho(w)$  by:  $\rho(w) = -z(w) + t(w)$ , recalling that  $\rho(w) \neq 0$  for any  $w$ . Then

$$I_{\Sigma^1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \rho^2(w) & 0 \\ 0 & 0 & \rho^2(w) \end{pmatrix}. \quad (6.78)$$

We can see that the first fundamental form is parametrized by one parameter variable. Also the first fundamental form of  $\Sigma^1(w, u, v)$  has signature  $(-, +, +)$  everywhere, which gives a Lorentz metric on  $\Sigma^1$ .

#### 6.4 Surface of Rotation Generated by Parabolic and Boost Subgroups

The surface of rotation of this case is generated by parabolic (null) and boost subgroups. So, we will use case two above. Suppose that the axis of rotations is given by  $l = (0, 1, 0, 0)$ .

In this case, we have an nonabelian subgroup of  $SO(3, 1)$ . So the matrices of rotation are given by  $\mathcal{M}_1.\mathcal{M}_3$ , and  $\mathcal{M}_3.\mathcal{M}_1$  since this case is not a commutative group.

Therefore, we can generate the surface of rotation in two distinct ways.

In both cases, as in previous section, we are interested in taking a planar "time-like" curve  $\gamma$  and rotating it with two dimensional sub-groups of isometry.

##### 6.4.1 Surface of Rotation Generated by $\mathcal{M}_1.\mathcal{M}_3$

The surface of rotation of this case is generated by  $N_x$  and  $B_z$ . Using  $\mathcal{M}_1.\mathcal{M}_3$ . By the same argument before, without loss of generality we assume that the curve  $\gamma$  lies in the  $yt$ -

plane. Hence , it can be parametrized by :

$$\gamma(w) = (0, y(w), 0, t(w)), \quad (6.79)$$

where  $y(w), t(w)$  are smooth functions. And the surface of rotation  $\Sigma^2$  can be parametrized by :

$$\Sigma^2(w, u, v) = \mathcal{M}_1(u) \cdot \mathcal{M}_3(v) \cdot \gamma(w), \quad (6.80)$$

or

$$\Sigma^2(w, u, v) = \begin{pmatrix} 1 & 0 & -u & u \\ 0 & 1 & 0 & 0 \\ u & 0 & 1 - \frac{u^2}{2} & \frac{u^2}{2} \\ u & 0 & -\frac{u^2}{2} & 1 + \frac{u^2}{2} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh(u) & \sinh(u) \\ 0 & 0 & \sinh(u) & \cosh(u) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ y(w) \\ 0 \\ t(w) \end{pmatrix} \quad (6.81)$$

Or,

$$\Sigma^2(w, u, v) = \begin{pmatrix} 1 & 0 & -u \cosh(v) + u \sinh(v) & -u \sinh(v) + u \cosh(v) \\ 0 & 1 & 0 & 0 \\ u & 0 & (1 - \frac{1}{2} u^2) \cosh(v) + \frac{1}{2} u^2 \sinh(v) & (1 - \frac{1}{2} u^2) \sinh(v) + \frac{1}{2} u^2 \cosh(v) \\ u & 0 & -\frac{1}{2} u^2 \cosh(v) + (1 + \frac{1}{2} u^2) \sinh(v) & -\frac{1}{2} u^2 \sinh(v) + (1 + \frac{1}{2} u^2) \cosh(v) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ y(w) \\ 0 \\ t(w) \end{pmatrix}. \quad (6.82)$$

So, the surface of rotation is given by :

$$\Sigma^2(w, u, v) = \begin{pmatrix} (-u \sinh(v) + u \cosh(v))t(w) \\ y(w) \\ ((1 - 1/2u^2) \sinh(v) + 1/2u^2 \cosh(v))t(w) \\ (-1/2u^2) \sinh(v) + (1 + 1/2u^2) \cosh(v))t(w) \end{pmatrix}. \quad (6.83)$$

Also,

$$\Sigma_w^2 = \begin{pmatrix} (-u \sinh(v) + u \cosh(v))t'(w) \\ y'(w) \\ ((1 - 1/2u^2) \sinh(v) + 1/2u^2 \cosh(v))t'(w) \\ (-1/2u^2) \sinh(v) + (1 + 1/2u^2) \cosh(v))t'(w) \end{pmatrix}. \quad (6.84)$$

and,

$$\Sigma_u^2 = \begin{pmatrix} (-\sinh(v) + \cosh(v))t(w) \\ 0 \\ (-u \sinh(v) + u \cosh(v))t(w) \\ (-u \sinh(v) + u \cosh(v))t(w) \end{pmatrix}. \quad (6.85)$$

Also,

$$\Sigma_v^2 = \begin{pmatrix} (-u \cosh(v) + u \sinh(v))t(w) \\ 0 \\ ((1 - 1/2u^2) \cosh(v) + 1/2u^2 \sinh(v))t(w) \\ (-1/2u^2) \cosh(v) + (1 + 1/2u^2) \sinh(v))t(w) \end{pmatrix}. \quad (6.86)$$

Now, recall the first fundamental form from (6.75).

We get:

$$\left\{ \begin{array}{l} g(\Sigma_w^2, \Sigma_w^2) = y'^2(w) - t'^2(w) \\ g(\Sigma_w^2, \Sigma_u^2) = 0 \\ g(\Sigma_w^2, \Sigma_v^2) = 0 \\ g(\Sigma_u^2, \Sigma_u^2) = t^2(w)(-\sinh(v) + \cosh(v))^2 = t^2(w)e^{-2v} \\ g(\Sigma_u^2, \Sigma_v^2) = 0 \\ g(\Sigma_v^2, \Sigma_v^2) = t^2(w). \end{array} \right. \quad (6.87)$$

Thus, the first fundamental form is given by :

$$I_{\Sigma^2} = \begin{pmatrix} y'^2(w) - t'^2(w) & 0 & 0 \\ 0 & t^2(w)e^{-2v} & 0 \\ 0 & 0 & t^2(w) \end{pmatrix}. \quad (6.88)$$

We also assume  $t(w) \neq 0$ , in order to obtain regular surface. And we assume that the curve  $\gamma$  is time-like, then we can assure that  $y'^2(w) - t'^2(w) = -1$ .

So we have:

$$I_{\Sigma^2} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & t^2(w)e^{-2v} & 0 \\ 0 & 0 & t^2(w) \end{pmatrix}. \quad (6.89)$$

This is the first fundamental form of this surface  $\Sigma^2(w, u, v)$ ; which has the signature of  $(-, +, +)$  everywhere, which also gives the Lorentz metric on  $\Sigma^2$ . This time the first fundamental form has two variable parameters.

#### 6.4.2 Surface of Rotation Generated by $\mathcal{M}_3.\mathcal{M}_1$

The surface of rotation of this case is generated by  $B_z$  and  $N_x$ . Using  $\mathcal{M}_3.\mathcal{M}_1$ . By the same argument before, without loss of generality we assume that the curve  $\gamma$  lies in the  $yt$ -plane. Hence, it can be parametrized by:

$$\gamma(w) = (0, y(w), 0, t(w)), \quad (6.90)$$

where  $y(w), t(w)$  are smooth functions, and  $t(w)$  is non-zero. Hence, the surface of rotation  $\Sigma^3$  can be parametrized by :

$$\Sigma^3(w, \beta, \alpha) = \mathcal{M}_3(\alpha).\mathcal{M}_1(\beta).\gamma(w), \quad (6.91)$$

or

$$\Sigma^3(w, \beta, \alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh(\alpha) & \sinh(\alpha) \\ 0 & 0 & \sinh(\alpha) & \cosh(\alpha) \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -\beta & \beta \\ 0 & 1 & 0 & 0 \\ \beta & 0 & 1 - \frac{\beta^2}{2} & \frac{\beta^2}{2} \\ \beta & 0 & -\frac{\beta^2}{2} & 1 + \frac{\beta^2}{2} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ y(w) \\ 0 \\ t(w) \end{pmatrix}. \quad (6.92)$$

Or,

$$\Sigma^3(w, \beta, \alpha) = \begin{pmatrix} 1 & 0 & -\beta & \beta \\ 0 & 1 & 0 & 0 \\ \cosh(\alpha)\beta + \sinh(\alpha)\beta & 0 & \cosh(\alpha)(1 - \frac{1}{2}\beta^2) - \frac{1}{2}\sinh(\alpha)\beta^2 & \frac{1}{2}\cosh(\alpha)\beta^2 + \sinh(\alpha)(1 + \frac{1}{2}\beta^2) \\ \cosh(\alpha)\beta + \sinh(\alpha)\beta & 0 & \sinh(\alpha)(1 - \frac{1}{2}\beta^2) - \frac{1}{2}\cosh(\alpha)\beta^2 & \frac{1}{2}\sinh(\alpha)\beta^2 + \cosh(\alpha)(1 + \frac{1}{2}\beta^2) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ y(w) \\ 0 \\ t(w) \end{pmatrix}. \quad (6.93)$$

So, the surface of rotation is given by :

$$\Sigma^3(w, \beta, \alpha) = \begin{pmatrix} \beta t(w) \\ y(w) \\ (1/2 \cosh(\alpha)\beta^2 + \sinh(\alpha)(1 + 1/2\beta^2)) t(w) \\ (1/2 \sinh(\alpha)\beta^2 + \cosh(\alpha)(1 + 1/2\beta^2)) t(w) \end{pmatrix}. \quad (6.94)$$

Also,

$$\Sigma_w^3 = \begin{pmatrix} \beta t'(w) \\ y'(w) \\ (1/2 \cosh(\alpha)\beta^2 + \sinh(\alpha)(1 + 1/2\beta^2)) t'(w) \\ (1/2 \sinh(\alpha)\beta^2 + \cosh(\alpha)(1 + 1/2\beta^2)) t'(w) \end{pmatrix}. \quad (6.95)$$

and,

$$\Sigma_\beta^3 = \begin{pmatrix} t(w) \\ 0 \\ (\cosh(\alpha)\beta + \sinh(\alpha)\beta) t(w) \\ (\cosh(\alpha)\beta + \sinh(\alpha)\beta) t(w) \end{pmatrix}. \quad (6.96)$$

Also,

$$\Sigma_\alpha^3 = \begin{pmatrix} 0 \\ 0 \\ (1/2 \sinh(\alpha) \beta^2 + \cosh(\alpha) (1 + 1/2 \beta^2)) t(w) \\ (1/2 \cosh(\alpha) \beta^2 + \sinh(\alpha) (1 + 1/2 \beta^2)) t(w) \end{pmatrix}. \quad (6.97)$$

Now, recall the first fundamental form from (6.75).

We get:

$$\left\{ \begin{array}{l} g(\Sigma_w^3, \Sigma_w^3) = y'^2(w) - t'^2(w) \\ g(\Sigma_w^3, \Sigma_\beta^3) = 0 \\ g(\Sigma_w^3, \Sigma_\alpha^3) = 0 \\ g(\Sigma_\beta^3, \Sigma_\beta^3) = t^2(w) \\ g(\Sigma_\beta^3, \Sigma_\alpha^3) = \beta t^2(w) \\ g(\Sigma_\alpha^3, \Sigma_\alpha^3) = t^2(w)(1 + \beta^2). \end{array} \right. \quad (6.98)$$

Thus, the first fundamental form is given by :

$$I_{\Sigma^3} = \begin{pmatrix} y'^2(w) - t'^2(w) & 0 & 0 \\ 0 & t^2(w) & \beta t^2(w) \\ 0 & \beta t^2(w) & t^2(w)(1 + \beta^2) \end{pmatrix}. \quad (6.99)$$

Now, we here assumed that the curve  $\gamma$  is time-like, so that  $y'^2(w) - t'^2(w) = -1$ .

Then we have:

$$I_{\Sigma^3} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & t^2(w) & \beta t^2(w) \\ 0 & \beta t^2(w) & t^2(w)(1 + \beta^2) \end{pmatrix}. \quad (6.100)$$

This is the first fundamental form of this surface  $\Sigma^3(w, \alpha, \beta)$ ; which has the signature of  $(-, +, +)$  everywhere, which gives the Lorentz metric on  $\Sigma^3$ . And the first fundamental form is a two variable parameter. Moreover, it is important to note that, the coordinates of parametrization are not orthogonal, since the first fundamental form is not diagonal.

### 6.4.3 The Relationship between the Parametrization of $\Sigma^2$ and $\Sigma^3$

The parametrization of  $\Sigma^2$  and  $\Sigma^3$  are generated by  $\mathcal{M}_1.\mathcal{M}_3$  and  $\mathcal{M}_3.\mathcal{M}_1$  respectively; which are a non-commutative sub-algebra isomorphic to the Lie algebra. They give the same surface of rotation. But with different parametrizations.

On equating those two cases above, we have :

$$\mathcal{M}_1(u).\mathcal{M}_3(v) = \mathcal{M}_3(\alpha).\mathcal{M}_1(\beta). \quad (6.101)$$

Then

$$\begin{aligned} & \begin{pmatrix} 1 & 0 & -u & u \\ 0 & 1 & 0 & 0 \\ u & 0 & 1 - \frac{u^2}{2} & \frac{u^2}{2} \\ u & 0 & -\frac{u^2}{2} & 1 + \frac{u^2}{2} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh(u) & \sinh(u) \\ 0 & 0 & \sinh(u) & \cosh(u) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh(\alpha) & \sinh(\alpha) \\ 0 & 0 & \sinh(\alpha) & \cosh(\alpha) \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -\beta & \beta \\ 0 & 1 & 0 & 0 \\ \beta & 0 & 1 - \frac{\beta^2}{2} & \frac{\beta^2}{2} \\ \beta & 0 & -\frac{\beta^2}{2} & 1 + \frac{\beta^2}{2} \end{pmatrix}. \end{aligned} \quad (6.102)$$

Or,

$$\begin{aligned} & \begin{pmatrix} 1 & 0 & -u \cosh(v) + u \sinh(v) & -u \sinh(v) + u \cosh(v) \\ 0 & 1 & 0 & 0 \\ u & 0 & (1 - \frac{1}{2}u^2) \cosh(v) + \frac{1}{2}u^2 \sinh(v) & (1 - \frac{1}{2}u^2) \sinh(v) + \frac{1}{2}u^2 \cosh(v) \\ u & 0 & -\frac{1}{2}u^2 \cosh(v) + (1 + \frac{1}{2}u^2) \sinh(v) & -\frac{1}{2}u^2 \sinh(v) + (1 + \frac{1}{2}u^2) \cosh(v) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & -\beta & \beta \\ 0 & 1 & 0 & 0 \\ \cosh(\alpha)\beta + \sinh(\alpha)\beta & 0 & \cosh(\alpha)(1 - \frac{1}{2}\beta^2) - \frac{1}{2}\sinh(\alpha)\beta^2 & \frac{1}{2}\cosh(\alpha)\beta^2 + \sinh(\alpha)(1 + \frac{1}{2}\beta^2) \\ \cosh(\alpha)\beta + \sinh(\alpha)\beta & 0 & \sinh(\alpha)(1 - \frac{1}{2}\beta^2) - \frac{1}{2}\cosh(\alpha)\beta^2 & \frac{1}{2}\sinh(\alpha)\beta^2 + \cosh(\alpha)(1 + \frac{1}{2}\beta^2) \end{pmatrix} \end{aligned} \quad (6.103)$$

Now, equating matrix entries, we get:

$$\alpha = v \quad , \quad \beta = ue^{-v}, \quad (6.104)$$

or

$$u = \beta e^\alpha \quad , \quad v = \alpha. \quad (6.105)$$

An explicit calculation verifies that :

$$\mathcal{M}_1(u).\mathcal{M}_3(v) = \mathcal{M}_3(v).\mathcal{M}_1(ue^{-v}). \quad (6.106)$$

### 6.5 Surface of Rotation Generated by Spherical Symmetric Case

This surface of rotation of this case is generated by case(4) above. We know the sphere in  $\mathbb{E}^3$  parametrized by :

$$\sigma(u, v) = (\cos(u) \sin(v), \cos(u) \cos(v), \sin(u)) \quad (6.107)$$

In the spherical symmetric of case (4) we therefore have the parametrization :

$$\Sigma^4(w, u, v) = \begin{pmatrix} \cos(u) \sin(v) z(w) \\ \cos(u) \cos(v) z(w) \\ \sin(u) z(w) \\ t(w) \end{pmatrix}. \quad (6.108)$$

Also,

$$\Sigma_w^4(w, u, v) = \begin{pmatrix} \cos(u) \sin(v) z'(w) \\ \cos(u) \cos(v) z'(w) \\ \sin(u) z'(w) \\ t'(w) \end{pmatrix}, \quad (6.109)$$



and,

$$\Sigma_u^4(w, u, v) = \begin{pmatrix} -\sin(u) \sin(v) z(w) \\ -\sin(u) \cos(v) z(w) \\ \cos(u) z(w) \\ 0 \end{pmatrix}, \quad (6.110)$$

also,

$$\Sigma_v^4(w, u, v) = \begin{pmatrix} \cos(u) \cos(v) z(w) \\ -\cos(u) \sin(v) z(w) \\ 0 \\ 0 \end{pmatrix}. \quad (6.111)$$

So again, one can calculate the first fundamental form of this surface. Recall the first fundamental form from (6.75).

We get:

$$\begin{cases} g(\Sigma_w^4, \Sigma_w^4) = z'^2(w) - t'^2(w) \\ g(\Sigma_w^4, \Sigma_u^4) = 0 \\ g(\Sigma_w^4, \Sigma_v^4) = 0 \\ g(\Sigma_u^4, \Sigma_u^4) = z^2(w) \\ g(\Sigma_u^4, \Sigma_v^4) = 0 \\ g(\Sigma_v^4, \Sigma_v^4) = \cos^2(u) z^2(w). \end{cases} \quad (6.112)$$

Thus, the first fundamental form is given by :

$$I_{\Sigma^4} = \begin{pmatrix} z'^2(w) - t'^2(w) & 0 & 0 \\ 0 & z^2(w) & 0 \\ 0 & 0 & \cos^2(u) z^2(w) \end{pmatrix}. \quad (6.113)$$

Now, if we assume that the generator is time-like . Then we can assure that  $z'^2(w) - t'^2(w) =$

–1. Then we have:

$$I_{\Sigma^4} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & z^2(w) & 0 \\ 0 & 0 & \cos^2(u)z^2(w) \end{pmatrix}. \quad (6.114)$$

In order to ensure that the surface is regular, we require  $z(w) \neq 0$  and  $t(w) \neq 0$ .

This is the first fundamental form of this surface  $\Sigma^4(w, u, v)$ ; which has the signature of  $(-, +, +)$  everywhere, which gives the Lorentz metric on  $\Sigma^4$ . Also we can observe that the first fundamental form has two variable parameters.

We will require another parametrization later in chapter 7 to find another conserved quantity, we choose

$$\sigma(\alpha, \beta) = (\sin(\alpha), \cos(\alpha) \cos(\beta), \cos(\alpha) \sin(\beta)) \quad (6.115)$$

given by rotation around the  $x$ -axis. This gives:

$$\Sigma^{4x} = \begin{pmatrix} \sin(\alpha) z(w) \\ \cos(\alpha) \cos(\beta) z(w) \\ \cos(\alpha) \sin(\beta) z(w) \\ t(w) \end{pmatrix}. \quad (6.116)$$

With the same calculation and argument again, one can compute the first fundamental form and can be given in final form by:

$$I_{\Sigma^{4x}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & z^2(w) & 0 \\ 0 & 0 & \cos^2(\alpha)z^2(w) \end{pmatrix}. \quad (6.117)$$

**The Relationship between the Parametrizations  $\Sigma^4$  and  $\Sigma^{4x}$** 

On equating the whole entries of the parametrization  $\Sigma^4(w, u, v)$  and  $\Sigma^{4x}(w, \alpha, \beta)$ , and then solving using *Maple* software we conclude that the relationship is given by:

$$u = \arcsin(\cos(\alpha) \sin(\beta)) \quad \text{and} \quad v = \arcsin\left(\frac{\sin(\alpha)}{\sqrt{1 - (\cos(\alpha))^2 (\sin(\beta))^2}}\right) \quad (6.118)$$

or

$$\alpha = \arcsin(\cos(u) \sin(v)) \quad \text{and} \quad \beta = \arcsin\left(\frac{\sin(u)}{\sqrt{1 - (\cos(u))^2 (\sin(v))^2}}\right). \quad (6.119)$$

**6.6 Surface of Rotation Generated by Boost and Rotation Subgroups**

The surface of rotation of this case is generated by a boost  $B_z$  and a rotation  $R_z$ . This matrix does not fix any axis, just the origin. We will use the case four to generate the surface of rotation. So, the matrices of rotations of this surface are  $\mathcal{M}_3, \mathcal{M}_4$ . We are interested in taking a planar curve  $\gamma$  and rotating it with two dimensional sub-groups of isometry.

As before, we can take the planar curve  $\gamma$  for this surface of rotation  $\Sigma^5(w, u, v)$  to be the intersection of  $\Sigma^5(w, u, v)$  with  $x = z = 0$ . So, without loss of generality we assume that the curve  $\gamma$  lies in the  $yt$ - plane if  $|y| > |t|$ . And if  $|y| < |t|$  we can take  $\gamma$  in the  $yz$ - plane, but as in the three dimensional case this gives the same result. So,  $\gamma$  can be parametrized by:

$$\gamma(w) = (0, y(w), 0, t(w)), \quad (6.120)$$

where  $y(w), t(w)$  are smooth functions. And  $y(w)$  is a positive function. The surface of rotation is denoted by  $\Sigma^5$ , and can be parametrized by :

$$\Sigma^5(w, u, v) = \mathcal{M}_3(u) \cdot \mathcal{M}_4(v) \cdot \gamma(w), \quad (6.121)$$

or

$$\Sigma^5(w, u, v) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh(u) & \sinh(u) \\ 0 & 0 & \sinh(u) & \cosh(u) \end{pmatrix} \cdot \begin{pmatrix} \cos(v) & -\sin(v) & 0 & 0 \\ \sin(v) & \cos(v) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ y(w) \\ 0 \\ t(w) \end{pmatrix}. \quad (6.122)$$

Or,

$$\Sigma^5(w, u, v) = \begin{pmatrix} \cos(v) & -\sin(v) & 0 & 0 \\ \sin(v) & \cos(v) & 0 & 0 \\ 0 & 0 & \cosh(u) & \sinh(u) \\ 0 & 0 & \sinh(u) & \cosh(u) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ y(w) \\ 0 \\ t(w) \end{pmatrix}. \quad (6.123)$$

So, the surface of rotation is given by :

$$\Sigma^5(w, u, v) = \begin{pmatrix} -\sin(v)y(w) \\ \cos(v)y(w) \\ \sinh(u)t(w) \\ \cosh(u)t(w) \end{pmatrix}. \quad (6.124)$$

Also,

$$\Sigma_w^5 = \begin{pmatrix} -\sin(v)y'(w) \\ \cos(v)y'(w) \\ \sinh(u)t'(w) \\ \cosh(u)t'(w) \end{pmatrix}, \quad (6.125)$$

and,

$$\Sigma_u^5 = \begin{pmatrix} 0 \\ 0 \\ \cosh(u)t(w) \\ \sinh(u)t(w) \end{pmatrix}, \quad (6.126)$$

Also,

$$\Sigma_v^5 = \begin{pmatrix} -\cos(v)y(w) \\ -\sin(v)y(w) \\ 0 \\ 0 \end{pmatrix}. \quad (6.127)$$

So, one can calculate the first fundamental form of this surface. Recall the first fundamental form from (6.75).

We get:

$$\begin{cases} g(\Sigma_w^5, \Sigma_w^5) = y'^2(w) - t'^2(w) \\ g(\Sigma_w^5, \Sigma_u^5) = 0 \\ g(\Sigma_w^5, \Sigma_v^5) = 0 \\ g(\Sigma_u^5, \Sigma_u^5) = t^2(w) \\ g(\Sigma_u^5, \Sigma_v^5) = 0 \\ g(\Sigma_v^5, \Sigma_v^5) = y^2(w). \end{cases} \quad (6.128)$$

Thus, the first fundamental form is given by :

$$I_{\Sigma^5} = \begin{pmatrix} y'^2(w) - t'^2(w) & 0 & 0 \\ 0 & t^2(w) & 0 \\ 0 & 0 & y^2(w) \end{pmatrix}. \quad (6.129)$$

Now, if we assume that the curve  $\gamma$  is time-like. Then we can assure that  $y'^2(w) - t'^2(w) =$

–1. Then we have:

$$I_{\Sigma^5} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & t^2(w) & 0 \\ 0 & 0 & y^2(w) \end{pmatrix}. \quad (6.130)$$

In order to ensure that the surface is regular, we require  $y(w) \neq 0$  and  $t(w) \neq 0$ .

This is the first fundamental form of this surface  $\Sigma^5(w, u, v)$ ; which has the signature of  $(-, +, +)$  everywhere, which gives the Lorentz metric on  $\Sigma^5$ . Also we can observe that the first fundamental form has one variable parameter.

### 6.7 Conclusion

In conclusion, in this chapter we found three different types of two dimensional sub-algebras. These generate two dimensional sub-groups of isometries, analogous to rotations in  $\mathbb{E}^3$ . These two dimensional sub-groups of isometries are used to parametrise three different types of surfaces of rotations ( one of the surfaces has two parametrizations) in  $\mathbb{M}^{3,1}$ .

These surfaces of rotations are called  $\Sigma^1, \Sigma^2, \Sigma^3, \Sigma^4$  and  $\Sigma^5$ . The surfaces which parametrized by  $\Sigma^1$  and  $\Sigma^5$  have one variable parameter of first fundamental form, with an orthonormal basis . As we will see, this means they are more amenable to generalizing Clairaut's theorem to them. However, the surface parametrized by  $\Sigma^2$  essentially on two variable parameter of first fundamental form with  $g(\Sigma_w^2, \Sigma_u^2) = g(\Sigma_w^2, \Sigma_v^2) = g(\Sigma_u^2, \Sigma_v^2) = 0$ , that means this surface has an orthonormal basis too, thus is possible to generate Clairaut's theorem to it, it does need some technique. But, the parametrization  $\Sigma^3$  has two variable parameters with no orthonormal basis on it. So Clairaut's theorem could not be applied to this parametrization of the surface in a straight forward way, as well as the surface parametrized by  $\Sigma^4$  of spherical case also Clairaut's theorem could not be applied in straight forward way because the two conserved quantities are not commute. Therefore, we will generalize Clairaut's theorem to three surfaces of rotation parametrized by  $\Sigma^1, \Sigma^2$  and  $\Sigma^5$ ; we will use  $\Sigma^3$  to help in  $\Sigma^2$  as they give the same surface of rotation. Also we will give a brief description of conserved quantities of  $\Sigma^4$ .



## 7. GENERALIZATION OF CLAIRAUT'S THEOREM TO 4D MINKOWSKI SPACE

In this chapter we take the surfaces of rotations  $\Sigma^1, \Sigma^2$  and  $\Sigma^5$  from chapter 6 and generalize Clairaut's theorem for these. Each section will take one surface of rotation and generalize Clairaut's theorem to it.

### 7.1 *Clairaut's Theorem of Surfaces of Rotations Generated by Two Parabolic Subgroups*

The surface of rotation is parametrized by two parabolic groups of matrices found in (6.71):

$$\Sigma^1(w, u, v) = \mathcal{M}_1(u) \cdot \mathcal{M}_2(v) \cdot \gamma(w) = \begin{pmatrix} -uz(w) + ut(w) \\ -vz(w) + vt(w) \\ \left(1 - \frac{v^2}{2} - \frac{u^2}{2}\right)z(w) + \left(\frac{v^2}{2} + \frac{u^2}{2}\right)t(w) \\ \left(-\frac{v^2}{2} - \frac{u^2}{2}\right)z(w) + \left(1 + \frac{v^2}{2} + \frac{u^2}{2}\right)t(w) \end{pmatrix}. \quad (7.1)$$

This has a first fundamental form given as in (6.77) by :

$$I_{\Sigma^1} = \begin{pmatrix} z'^2(w) - t'^2(w) & 0 & 0 \\ 0 & (-z(w) + t(w))^2 & 0 \\ 0 & 0 & (-z(w) + t(w))^2 \end{pmatrix}. \quad (7.2)$$



We assume that  $z'^2(w) - t'^2(w) = -1$  and  $-z(w) + t(w) = \rho(w)$ ,  $\rho(w) > 0$ . Then

$$I_{\Sigma^1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \rho^2(w) & 0 \\ 0 & 0 & \rho^2(w) \end{pmatrix}. \quad (7.3)$$

Therefore, the first fundamental form is given by:

$$-d^2w + \rho^2(w)d^2u + \rho^2(w)d^2v \quad (7.4)$$

From the fundamental form, we have the Lagrangian equation:

$$-\dot{w}^2 + \rho^2(w)\dot{u}^2 + \rho^2(w)\dot{v}^2 \quad (7.5)$$

giving Euler-Lagrange equations

$$\begin{aligned} \ddot{w} &= -\rho(w)\rho'(w)(\dot{u}^2 + \dot{v}^2) \\ \frac{d}{ds}(\rho^2(w)\dot{u}) &= 0 \\ \frac{d}{ds}(\rho^2(w)\dot{v}) &= 0 \end{aligned} \quad (7.6)$$

Now, let  $\gamma(s)$  be a time-like geodesic on the surface of rotation, so it is given by  $w(s), u(s), v(s)$ .

Then, we can see that

$$\dot{\gamma}(s) = \dot{w}\Sigma_w^1 + \dot{u}\Sigma_u^1 + \dot{v}\Sigma_v^1. \quad (7.7)$$

We notice that,  $\Sigma_w^1 = n_w$  is a unit time-like vector pointing along the meridians. And  $\Sigma_u^1 = \rho(w)n_u$  where  $n_u$  is a unit space-like vector pointing along the  $u$ -axis of the parallels. Also  $\Sigma_v^1 = \rho(w)n_v$  where  $n_v$  is a unit space-like vector pointing along the  $v$ -axis of the parallels. It also follows that the plane spanned by  $n_u$  and  $n_v$  is space-like, with  $n_u$  and  $n_v$  providing an orthonormal basis. Furthermore,  $g(\Sigma_w^1, \Sigma_u^1) = g(\Sigma_w^1, \Sigma_v^1) = g(\Sigma_u^1, \Sigma_v^1) = 0$ ; so we have an orthonormal basis.

Therefore,

$$\dot{\gamma}(s) = \dot{w}n_w + \dot{u}\rho(w)n_u + \dot{v}\rho(w)n_v. \quad (7.8)$$

However, if  $\dot{u}\rho(w)n_u + \dot{v}\rho(w)n_v = 0$ , then  $\dot{u} = \dot{v} = 0$ , so  $\dot{w} = 1$ , and so the Euler-Lagrangian equations are satisfied, hence:

$$\dot{\gamma}(s) = \dot{w}n_w; \quad (7.9)$$

so, all meridians are geodesics.

we consider the case

$$\dot{u}\rho(w)n_u + \dot{v}\rho(w)n_v \neq 0, \quad (7.10)$$

and define  $n_{w^\perp}$  by :

$$n_{w^\perp} = \frac{\dot{u}\rho(w)n_u + \dot{v}\rho(w)n_v}{\|\dot{u}\rho(w)n_u + \dot{v}\rho(w)n_v\|} \quad (7.11)$$

this is a unit vector perpendicular to  $n_w$ .

Then in Minkowski setting we have

$$\dot{\gamma}(s) = n_w \cosh(\theta) + n_{w^\perp} \sinh(\theta), \quad (7.12)$$

such that,  $n_{w^\perp}$  is a unit vector perpendicular to  $n_w$  in the surface of rotation, which has  $n_w$  as a unit vector along the meridians.

If  $\phi$  is the angle between  $n_u$  and  $n_{w^\perp}$  then:

$$\dot{\gamma}(s) = n_w \cosh(\theta) + [n_u \cos(\phi) + n_v \sin(\phi)] \sinh(\theta). \quad (7.13)$$

From equations (7.8) and (7.13), we have :

$$\rho(w)\dot{u} = \cos(\phi) \sinh(\theta) \text{ and } \rho(w)\dot{v} = \sin(\phi) \sinh(\theta) \quad (7.14)$$

gives

$$\rho^2(w)\dot{u} = \rho(w) \cos(\phi) \sinh(\theta) \text{ and } \rho(w)^2\dot{v} = \rho(w) \sin(\phi) \sinh(\theta) \quad (7.15)$$

We can conclude that the second and third Euler-Lagrangian equations are equivalent to  $\rho(w) \cos \phi \sinh(\theta)$  and  $\rho(w) \sin \phi \sinh(\theta)$  being constants.

Moreover, if we constrain one parameter, say  $v = k$ , where  $k$  is constant. Then from (7.15) we get  $\phi = 0$ , so then  $\rho(w) \sinh(\theta)$  is constant. Also if we restrict  $u = k$ . Then  $\phi = \frac{\pi}{2}$ , so  $\rho(w) \sinh(\theta)$  is again constant. This discussion shows how the case of a null rotation in  $\mathbb{M}^{2,1}$  given in (chapter 5) is embedded in more general case.

Conversely, let  $\gamma$  be a proper parametrized time-like geodesic curve, such that  $\rho(w) \cos \phi \sinh(\theta) = \rho^2(w)\dot{u}$  and  $\rho(w) \sin \phi \sinh(\theta) = \rho^2(w)\dot{v}$  are constants, and  $\dot{w} \neq 0$ .

So,

$$\dot{w}^2 - \rho^2(w)(\dot{u}^2 + \dot{v}^2) = 1 \quad \text{where} \quad \rho^2(w)\dot{u}, \rho^2(w)\dot{v} \text{ are constants}$$

Differentiating this with respect to  $s$ , we have :

$$2\dot{w}\ddot{w} - 2\rho(w)\rho'(w)\dot{w}(\dot{u}^2 + \dot{v}^2) - 2\rho^2(w)(\dot{u}\ddot{u} + \dot{v}\ddot{v}) = 0 \quad (7.16)$$

Or

$$\dot{w}\ddot{w} - \rho(w)\rho'(w)\dot{w}(\dot{u}^2 + \dot{v}^2) - \rho^2(w)(\dot{u}\ddot{u} + \dot{v}\ddot{v}) = 0 \quad (7.17)$$

Now, from:

$$\frac{d}{ds}(\rho^2(w)\dot{u}) = 0 = 2\rho(w)\rho'(w)\dot{w}\dot{u} + \rho^2(w)\ddot{u} \quad (7.18a)$$

$$\frac{d}{ds}(\rho^2(w)\dot{v}) = 0 = 2\rho(w)\rho'(w)\dot{w}\dot{v} + \rho^2(w)\ddot{v} \quad (7.18b)$$

Now, multiplying (7.18a) by  $\dot{u}$ , and (7.18b) by  $\dot{v}$ , then add them together, we obtain:

$$2\rho(w)\rho'(w)\dot{w}(\dot{u}^2 + \dot{v}^2) + \rho^2(w)(\dot{u}\ddot{u} + \dot{v}\ddot{v}) = 0. \quad (7.19)$$

Adding (7.17) to (7.19) gives:

$$\dot{w}\ddot{w} + \rho(w)\rho'(w)\dot{w}(\dot{u}^2 + \dot{v}^2) = 0. \quad (7.20)$$

And,  $\dot{w} \neq 0$ , so

$$\ddot{w} = -\rho(w)\rho'(w)(\dot{u}^2 + \dot{v}^2) \quad (7.21)$$

Which is the first Euler-Lagrangian equation. It follows that  $\gamma(s)$  is time-like geodesic.

□

Now, one can define the Hamiltonian version of the Lagrangian equation. So recalling equations (7.5) we have Lagrangian equation given by:

$$L = \frac{1}{2} (-\dot{w}^2 + \rho^2(w)\dot{u}^2 + \rho^2(w)\dot{v}^2) \quad (7.22)$$

and the partial derivatives of all components of  $L$  are

$$\frac{\partial L}{\partial \dot{w}} = P_w = -\dot{w} \quad , \quad \frac{\partial L}{\partial \dot{u}} = P_u = \rho^2(w)\dot{u} \quad \text{and} \quad \frac{\partial L}{\partial \dot{v}} = P_v = \rho^2(w)\dot{v} \quad (7.23)$$

Also from equation (7.6) we have  $P_u$  and  $P_v$  are constants.

And,

$$\dot{w} = -P_w \quad , \quad \dot{u} = P_u/\rho^2 \quad \text{and} \quad \dot{v} = P_v/\rho^2 \quad (7.24)$$

Now, let us recall the Hamiltonian equation given in (2.32) by:

$$H(q, p) = \sum_{\beta=1}^n p_{\beta} \dot{q}_{\beta}(q, p) - L(q, \dot{q}(q, p)). \quad (7.25)$$

Then for this case it is :

$$H = P_w \dot{w} + P_u \dot{u} + P_v \dot{v} - \frac{1}{2} (-\dot{w}^2 + \rho^2(w)\dot{u}^2 + \rho^2(w)\dot{v}^2) \quad (7.26)$$

substituting using (7.24) we obtain

$$H = \frac{1}{2} \left( -P_w + \frac{P_u^2}{\rho^2(w)} + \frac{P_v^2}{\rho^2(w)} \right). \quad (7.27)$$

We know that the Hamiltonian function commutes by Poisson bracket (2.34) with all conserved quantities. Therefore, we need just proof that the two conserved quantities of this Hamiltonian function  $P_u$  and  $P_v$  commute with each other.

So, by using the Poisson bracket which given in (2.34) by:

$$[f, g] = \sum_{\alpha=1}^n \left( \frac{\partial f}{\partial q_\alpha} \frac{\partial g}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha} \frac{\partial g}{\partial q_\alpha} \right). \quad (7.28)$$

Then, at this time  $f = P_u = \rho^2 \dot{u}$  and  $g = P_v = \rho^2 \dot{v}$  we obtain

$$\begin{aligned} [P_u, P_v] &= \frac{\partial P_u}{\partial u} \frac{\partial P_v}{\partial P_u} + \frac{\partial P_u}{\partial v} \frac{\partial P_v}{\partial P_v} - \frac{\partial P_u}{\partial P_u} \frac{\partial P_v}{\partial u} - \frac{\partial P_u}{\partial P_v} \frac{\partial P_v}{\partial v} \\ &= (0)(0) + (0)(1) - (1)(0) - (0)(0) = 0. \end{aligned} \quad (7.29)$$

Thus, the conserved quantities commute, then the system is integrable. So the geodesics are in terms of integrals and solutions of algebraic equations.

Hence, as in the example (2.13) we have the Lagrangian:

$$L = \frac{1}{2} (-\dot{w}^2 + \rho^2(w) \dot{u}^2 + \rho^2(w) \dot{v}^2) \quad (7.30)$$

with two constants:

$$\frac{d}{ds}(\rho^2(w) \dot{u}) = 0 \quad \text{and} \quad \frac{d}{ds}(\rho^2(w) \dot{v}) = 0 \quad (7.31)$$

then,

$$\rho^2(w) \dot{u} = \Omega_1 \quad \text{and} \quad \rho^2(w) \dot{v} = \Omega_2, \quad (7.32)$$

where  $\Omega_1$  and  $\Omega_2$  are constants.

So,

$$\dot{u} = \Omega_1/\rho^2(w) \quad \text{and} \quad \dot{v} = \Omega_2/\rho^2(w) \quad (7.33)$$

and so,

$$L = \frac{1}{2} \left( -\dot{w}^2 + \frac{\Omega_1^2}{\rho^2(w)} + \frac{\Omega_2^2}{\rho^2(w)} \right) \quad (7.34)$$

giving

$$\dot{w}^2 = \frac{\Omega_1^2}{\rho^2(w)} + \frac{\Omega_2^2}{\rho^2(w)} - 2L \quad (7.35)$$

or

$$\dot{w} = \sqrt{\frac{\Omega_1^2}{\rho^2(w)} + \frac{\Omega_2^2}{\rho^2(w)} - 2L} \quad (7.36)$$

rearranging this

$$\frac{dw}{\sqrt{\frac{\Omega_1^2}{\rho^2(w)} + \frac{\Omega_2^2}{\rho^2(w)} - 2L}} = ds \quad (7.37)$$

so that

$$s = \int \frac{dw}{\sqrt{\frac{\Omega_1^2}{\rho^2(w)} + \frac{\Omega_2^2}{\rho^2(w)} - 2L}} + C_1 \quad (7.38)$$

This specifies  $w$  as a function of  $s$

we now return to

$$\dot{u} = \Omega_1/\rho^2(w) \quad \text{and} \quad \dot{v} = \Omega_2/\rho^2(w) \quad (7.39)$$

and  $w$  is a function of  $s$  obtained above. Then

$$\dot{u} = \Omega_1/\rho^2(w(s)) \quad \text{and} \quad \dot{v} = \Omega_2/\rho^2(w(s)) \quad (7.40)$$

so that

$$u = \int \frac{\Omega_1}{\rho^2(w(s))} ds + C_2 \quad \text{and} \quad v = \int \frac{\Omega_2}{\rho^2(w(s))} ds + C_3 \quad (7.41)$$

which give all  $w, u$  and  $v$  explicitly in terms of integrals (quadratures).

## 7.2 Clairaut's Theorem of Surfaces of Rotations Generated by Parabolic and Boost Subgroups

In this section we study the surface of rotation parametrized by  $\Sigma^2$  and  $\Sigma^3$ . First with  $\Sigma^2$  we generate the conserved quantities then Clairatu's theorem, and then shown that these conserved quantities are not commute. Therefore, geodesics equations are not given in terms of integrals. After that the parametrization  $\Sigma^3$  which does not have an orthonormal basis of the first fundamental form, we here describe the conserved quantities, and then discuss that these conserved quantities are not commute; again the geodesics equations can not be given in terms of integrals.

### 7.2.1 Surfaces of Rotations Parametrized by $\Sigma^2$

Recall the surface of rotation parametrized by rotation and parabolic given in chapter six equation (6.83) for the case of the time-like generator:

$$\Sigma^2(w, u, v) = \mathcal{M}_1(u) \cdot \mathcal{M}_3(v) \cdot \gamma(w) = \begin{pmatrix} (-u \sinh(v) + u \cosh(v))t(w) \\ y(w) \\ ((1 - 1/2u^2) \sinh(v) + 1/2u^2 \cosh(v))t(w) \\ (-1/2u^2) \sinh(v) + (1 + 1/2u^2) \cosh(v))t(w) \end{pmatrix}, \quad (7.42)$$

and this first fundamental form is given by (6.88):

$$I_{\Sigma^2} = \begin{pmatrix} y'^2(w) - t'^2(w) & 0 & 0 \\ 0 & t^2(w)e^{-2v} & 0 \\ 0 & 0 & t^2(w) \end{pmatrix}. \quad (7.43)$$

Now, if we apply the condition of  $y'^2(w) - t'^2(w) = -1$ , then we have:

$$I_{\Sigma^2} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & t^2(w)e^{-2v} & 0 \\ 0 & 0 & t^2(w) \end{pmatrix}. \quad (7.44)$$

So,

$$-d^2w + t^2(w)e^{-2v}d^2u + t^2(w)d^2v. \quad (7.45)$$

From the first fundamental form, we have the Lagrangian given by:

$$-\dot{w}^2 + t^2(w)e^{-2v}\dot{u}^2 + t^2(w)\dot{v}^2. \quad (7.46)$$

A geodesic on the surface is given by the Euler Lagrangian Equation:

$$\frac{d}{ds} \left( \frac{\partial L}{\partial \dot{w}} \right) = \frac{\partial L}{\partial w}, \quad (7.47)$$

which gives :

$$\ddot{w} = -t(w)t'(w)e^{-2v}\dot{u}^2 - t(w)t'(w)\dot{v}^2. \quad (7.48)$$

Also,

$$\frac{d}{ds} \left( \frac{\partial L}{\partial \dot{u}} \right) = \frac{\partial L}{\partial u} = 0, \quad (7.49)$$

which means:

$$\frac{\partial L}{\partial \dot{u}} = 2t^2(w)e^{-2v}\dot{u} \quad (7.50)$$

is constant along the geodesic

And,

$$\frac{d}{ds} \left( \frac{\partial L}{\partial \dot{v}} \right) = \frac{\partial L}{\partial v} = -2t^2(w)e^{-2v}\dot{u}^2 \neq 0, \quad (7.51)$$

So,

$$\frac{\partial L}{\partial \dot{v}} \quad (7.52)$$



is not constant

Now, we should seek a constant for the second Euler-Lagrangian. Then we change the coordinate of this equation into  $\alpha, \beta$  terms, as defined in chapter 6 (6.4.3).

The relationship between  $u, v$  terms and  $\alpha, \beta$  terms is given by :

$$\alpha = v \quad , \quad \beta = ue^{-v}, \quad (7.53)$$

or

$$u = \beta e^\alpha \quad , \quad v = \alpha. \quad (7.54)$$

Thus the Lagrangian now is given by:

$$-\dot{w}^2 + t^2(w)\dot{\beta}^2 + 2t^2(w)\beta\dot{\beta}\dot{\alpha} + t^2(w)(1 + \beta^2)\dot{\alpha}^2. \quad (7.55)$$

And, for the constant related to  $v$ , we will consider the Euler-Lagrangian equation for  $\alpha$ .

So,

$$\frac{d}{ds} \left( \frac{\partial L}{\partial \dot{\alpha}} \right) = \frac{\partial L}{\partial \alpha} = 0. \quad (7.56)$$

Then

$$\frac{\partial L}{\partial \dot{\alpha}} = 2t^2(w)\beta\dot{\beta} + 2t^2(w)(1 + \beta^2)\dot{\alpha} \quad (7.57)$$

is constant.

Now , we express this constant of  $\alpha, \beta$  terms into  $u, v$  terms.

So

$$\frac{\partial L}{\partial \dot{\alpha}} = u(2t^2(w)e^{-2v}\dot{u}) + 2t^2(w)\dot{v} \quad (7.58)$$

is constant.

Here, the term  $u(2t^2(w)e^{-2v}\dot{u}) + 2t^2(w)\dot{v}$  should be independent of  $s$  . We confirm this as follows:

First,

$$\frac{d}{ds} \left( \frac{\partial L}{\partial \dot{v}} \right) = \frac{\partial L}{\partial v} \quad (7.59)$$

and

$$\frac{\partial L}{\partial \dot{v}} = 2t^2(w)\dot{v} \quad \text{and} \quad \frac{\partial L}{\partial v} = -2t^2(w)e^{-2v}\dot{u}^2 \quad (7.60)$$

so,

$$\frac{d}{ds}(2t^2(w)\dot{v}) = -2t^2(w)e^{-2v}\dot{u}^2 \quad (7.61)$$

or

$$4t(w)t'(w)\dot{v} + 2t^2(w)\ddot{v} = -2t^2(w)e^{-2v}\dot{u}^2. \quad (7.62)$$

On the other hand, let's differentiate  $u(2t^2(w)e^{-2v}\dot{u}) + 2t^2(w)\dot{v}$  with respect to  $s$ :

$$\frac{d}{ds}(u(2t^2(w)e^{-2v}\dot{u}) + 2t^2(w)\dot{v}) = 2t^2(w)e^{-2v}\dot{u}^2 + u\frac{d}{ds}(2t^2(w)e^{-2v}\dot{u}) + 4t(w)t'(w)\dot{v} + 2t^2(w)\ddot{v}. \quad (7.63)$$

Then, using (7.50) and (7.62) yields;

$$\frac{d}{ds}(u(2t^2(w)e^{-2v}\dot{u}) + 2t^2(w)\dot{v}) = 0. \quad (7.64)$$

Then,

$$u(2t^2(w)e^{-2v}\dot{u}) + 2t^2(w)\dot{v} \quad (7.65)$$

is constant.

It follows that the time-like geodesics are given by :

$$\begin{aligned} \ddot{v} &= -t(w)t'(w)e^{-2v}\dot{u}^2 - t(w)t'(w)\dot{v}^2, \\ \frac{d}{ds}(t^2(w)e^{-2v}\dot{u}) &= 0 \\ \frac{d}{ds}(u(t^2(w)e^{-2v}\dot{u}) + t^2(w)\dot{v}) &= 0. \end{aligned} \quad (7.66)$$

Now, let  $\gamma(s)$  be a curve on  $\Sigma^2$ , and assume that  $\gamma(s)$  is time-like geodesic on the surface of rotation  $\Sigma^2$ , so it is given by :  $w(s), u(s), v(s)$  .

Thus,

$$\dot{\gamma}(s) = \dot{w}\Sigma_w^2 + \dot{u}\Sigma_u^2 + \dot{v}\Sigma_v^2. \quad (7.67)$$

As before, we can note that  $\Sigma_w^2 = n_w$  is a unit time-like vector pointing along the meridians. And  $\Sigma_u^2 = t(w)e^{-v}n_u$ , where  $n_u$  is a unit space-like vector pointing along the  $u$ -axis of the parallels. Further,  $\Sigma_v^2 = t(w)n_v$  where  $n_v$  is a unit space-like vector pointing along the  $v$ -axis of the parallels. It also follows that the plane spanned by  $n_u$  and  $n_v$  is space-like, with  $n_u$  and  $n_v$  providing an orthonormal basis. Furthermore,  $g(\Sigma_w^2, \Sigma_u^2) = g(\Sigma_w^2, \Sigma_v^2) = g(\Sigma_u^2, \Sigma_v^2) = 0$ . So, we have an orthonormal basis.

Therefore,

$$\dot{\gamma}(s) = \dot{w}n_w + \dot{u}t(w)e^{-v}n_u + \dot{v}t(w)n_v \quad (7.68)$$

we now define a vector  $n_{w^\perp}$  by :

$$n_{w^\perp} = \frac{\dot{u}t(w)e^{-v}n_u + \dot{v}t(w)n_v}{\|\dot{u}t(w)e^{-v}n_u + \dot{v}t(w)n_v\|} \quad (7.69)$$

give a unit vector perpendicular to  $n_w$ .

For the time-like vector  $t$  tangent to the surface can be written as

$$n_w \cosh(\theta) + n_{w^\perp} \sinh(\theta), \quad (7.70)$$

where  $n_{w^\perp}$  is the unit vector perpendicular to  $n_w$ . And  $\theta$  is the hyperbolic angle between  $\dot{\gamma}$  and  $n_w$ .

So, in Minkowski, the curve  $\gamma(s)$  can be given by :

$$\dot{\gamma}(s) = n_w \cosh(\theta) + n_{w^\perp} \sinh(\theta). \quad (7.71)$$

Furthermore, if  $\phi$  is the angle between  $n_u$  and  $n_{w^\perp}$ , then;

$$\dot{\gamma}(s) = n_w \cosh(\theta) + [n_u \cos(\phi) + n_v \sin(\phi)] \sinh(\theta) \quad (7.72)$$

From equations (7.68) and (7.72). We have;

$$\dot{u}t(w)e^{-v} = \cos(\phi) \sinh(\theta). \quad (7.73)$$

Which gives

$$\dot{u}t^2(w)e^{-2v} = t(w)e^{-v} \cos(\phi) \sinh(\theta) \quad (7.74)$$

is constant.

And

$$\dot{u} = \frac{e^v \cos(\phi) \sinh(\theta)}{t(w)}. \quad (7.75)$$

Also

$$\dot{v}t(w) = \sin(\phi) \sinh(\theta). \quad (7.76)$$

Which gives

$$\dot{v}t^2(w) = t(w) \sin(\phi) \sinh(\theta). \quad (7.77)$$

Or

$$u(t^2(w)e^{-2v}\dot{u}) + \dot{v}t^2(w) = u(t^2(w)e^{-2v}\dot{u}) + t(w) \sin(\phi) \sinh(\theta) \quad (7.78)$$

is constant.

Therefore, from (7.75), we obtain

$$u(t^2(w)e^{-2v}\dot{u}) + \dot{v}t^2(w) = ut(w)e^{-v} \cos(\phi) \sinh(\theta) + t(w) \sin(\phi) \sinh(\theta) \quad (7.79)$$

is constant.

As before, we can restrict one variable by taking  $v = k$  where  $k$  is constant. From (7.79) we get  $\phi = 0$ , then the equation (7.74) becomes  $k_1 t(w) \sinh(\theta)$  is constant, where

$k_1 = e^{-k}$ . Also if we restrict  $u = k$ , from (7.74) we get  $\phi = \frac{\pi}{2}$ , so the equation (7.79) satisfies  $t(w) \sinh(\theta)$  is again constant. This discussion recovers the case of boosts and null rotation in  $\mathbb{M}^{2,1}$ .

We can see then, the second and third Euler-Lagrangian equations are equivalent to  $t(w)e^{-v} \cos(\phi) \sinh(\theta)$  and  $ut(w)e^{-v} \cos(\phi) \sinh(\theta) + t(w) \sin(\phi) \sinh(\theta)$  being constants.

Conversely, let  $\gamma$  be a proper time parametrized geodesic curve, such that  $t(w)e^{-v} \cos \phi \sinh(\theta) = t^2(w)e^{-2v}\dot{u}$  and  $ut(w)e^{-v} \cos(\phi) \sinh(\theta) + t(w) \sin(\phi) \sinh(\theta) = u(t^2(w)e^{-2v}\dot{u}) + t^2(w)\dot{v}$  are constants, and  $\dot{w} \neq 0$ .

So,

$$\dot{w}^2 - (t^2(w)e^{-2v}\dot{u}^2 + t^2(w)\dot{v}^2) = 1 \quad \text{and} \quad t^2(w)e^{-2v}\dot{u}, u(t^2(w)e^{-2v}\dot{u}) + t^2(w)\dot{v} \text{ are constants.}$$

Differentiating this with respect to  $s$ , we have :

$$2\dot{w}\ddot{w} - 2tt'\dot{w}e^{-2v}\dot{u}^2 + 2t^2e^{-2v}\dot{v}\dot{u}^2 - 2t^2e^{-2v}\dot{u}\ddot{u} - 2tt'\dot{w}\dot{v}^2 - 2t^2\dot{v}\ddot{v} = 0, \quad (7.80)$$

or

$$\dot{w}\ddot{w} - tt'\dot{w}e^{-2v}\dot{u}^2 + t^2e^{-2v}\dot{v}\dot{u}^2 - t^2e^{-2v}\dot{u}\ddot{u} - tt'\dot{w}\dot{v}^2 - t^2\dot{v}\ddot{v} = 0. \quad (7.81)$$

Now, from:

$$\frac{d}{ds}(t^2(w)e^{-2v}\dot{u}) = 0 = 2tt'\dot{w}e^{-2v}\dot{u} - 2t^2e^{-2v}\dot{v}\dot{u} + t^2e^{-2v}\ddot{u} \quad (7.82a)$$

$$\frac{d}{ds}(u(t^2(w)e^{-2v}\dot{u}) + t^2(w)\dot{v}) = 0 = t^2e^{-2v}\dot{u}^2 + u \left[ \frac{d}{ds}(t^2(w)e^{-2v}\dot{u}) \right] + 2tt'\dot{w}\dot{v} + t^2\ddot{v} \quad (7.82b)$$

Since  $\frac{d}{ds}(t^2(w)e^{-2v}\dot{u}) = 0$ , so the equation (7.82b) becomes:

$$\frac{d}{ds}(u(t^2(w)e^{-2v}\dot{u}) + t^2(w)\dot{v}) = 0 = t^2e^{-2v}\dot{u}^2 + 2tt'\dot{w}\dot{v} + t^2\ddot{v} \quad (7.83)$$

Now, multiplying (7.82a) by  $\dot{u}$ , and (7.83) by  $\dot{v}$ , then adding them together, we obtain:

$$2tt'\dot{w}e^{-2v}\dot{u}^2 + 2tt'\dot{w}\dot{v}^2 - t^2e^{-2v}\dot{u}\dot{u}^2 + t^2e^{-2v}\dot{u}\ddot{u} + t^2\dot{v}\ddot{v} = 0 \quad (7.84)$$

then, adding (7.81) to (7.84) gives:

$$\dot{w}\ddot{w} + t(w)t'(w)\dot{w}e^{-2v}\dot{u}^2 + t(w)t'(w)\dot{w}\dot{v}^2 = 0, \quad (7.85)$$

or

$$\dot{w}\ddot{w} = -t(w)t'(w)\dot{w}e^{-2v}\dot{u}^2 - t(w)t'(w)\dot{w}\dot{v}^2. \quad (7.86)$$

And,  $\dot{w} \neq 0$ , so

$$\ddot{w} = -t(w)t'(w)e^{-2v}\dot{u}^2 - t(w)t'(w)\dot{v}^2. \quad (7.87)$$

Which is the first Euler-Lagrangian equation. It follows that  $\gamma(s)$  is time-like geodesic.

□

Now, as in previous section we define the Hamiltonian function of the Lagrangian equation. So recall the Lagrangian equation of this surface:

$$L = \frac{1}{2} (-\dot{w}^2 + t^2(w)e^{-2v}\dot{u}^2 + t^2(w)\dot{v}^2) \quad (7.88)$$

and the partial derivatives of all components of  $L$  are

$$\frac{\partial L}{\partial \dot{w}} = P_w = -\dot{w} \quad , \quad \frac{\partial L}{\partial \dot{u}} = P_u = t^2e^{-2v}\dot{u} \quad \text{and} \quad \frac{\partial L}{\partial \dot{v}} = P_v = t^2(w)\dot{v} \quad (7.89)$$

So,

$$\dot{w} = -P_w \quad , \quad \dot{u} = \frac{P_u}{t^2e^{-2v}} \quad \text{and} \quad \dot{v} = P_v/t^2 \quad (7.90)$$

Now, we have the Hamiltonian function for this case:

$$H = P_w \dot{w} + P_u \dot{u} + P_v \dot{v} - \frac{1}{2} (-\dot{w}^2 + t^2(w) e^{-2v} \dot{u}^2 + t^2(w) \dot{v}^2) \quad (7.91)$$

substituting using (7.90) we obtain

$$H = \frac{1}{2} \left( -P_w + \frac{P_u^2}{t^2 e^{-2v}} + \frac{P_v^2}{t^2(w)} \right). \quad (7.92)$$

This system has the conserved quantities given in (7.66) by:

$$P_u = t^2(w) e^{-2v} \dot{u} \quad \text{and} \quad P_{uv} = u(t^2(w) e^{-2v} \dot{u}) + t^2(w) \dot{v} \quad (7.93)$$

or

$$P_u = t^2(w) e^{-2v} \dot{u} \quad \text{and} \quad P_{uv} = u P_u + P_v \quad (7.94)$$

We need to show that either these conserved quantities commute or not. This to ensure that the system can be produced by Liouville Arnol'd theorem into integrable system or not.

So, by using the Poisson bracket which given in (2.34) by:

$$[f, g] = \sum_{\alpha=1}^n \left( \frac{\partial f}{\partial q_\alpha} \frac{\partial g}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha} \frac{\partial g}{\partial q_\alpha} \right). \quad (7.95)$$

Then, at this time  $f = P_u = t^2(w) e^{-2v} \dot{u}$  and  $g = P_{uv} = u(t^2(w) e^{-2v} \dot{u}) + t^2(w) \dot{v}$  we obtain

$$\begin{aligned} [P_u, P_{uv}] &= \frac{\partial P_u}{\partial u} \frac{\partial P_{uv}}{\partial P_u} + \frac{\partial P_u}{\partial v} \frac{\partial P_{uv}}{\partial P_{uv}} - \frac{\partial P_u}{\partial P_u} \frac{\partial P_{uv}}{\partial u} - \frac{\partial P_u}{\partial P_{uv}} \frac{\partial P_{uv}}{\partial v} \\ &= (0)(u) + (-2t^2(w) e^{-2v} \dot{u})(1) - (1)(t^2(w) e^{-2v} \dot{u}) - (0)(1) \\ &= -3t^2(w) e^{-2v} \dot{u} \neq 0. \end{aligned} \quad (7.96)$$

Thus, the conserved quantities do not commute, as a result the system is not integrable. So the geodesics can not be expressed in terms of integrals.

In conclusion, we still have the conserved quantities, but in this case they depend explicitly on  $u$  and  $v$  as well as on  $w$ , so we do not have the same interpretation as analogues

of angular momentum in both. Only in the first case does this held.

### 7.2.2 Surfaces of Rotations Parametrized by $\Sigma^3$

Recall the surface of rotation of this case which given in chapter six equation (6.94) by

$$\Sigma^3(w, \beta, \alpha) = \mathcal{M}_3(\alpha) \cdot \mathcal{M}_1(\beta) \cdot \gamma(w) = \begin{pmatrix} \beta t(w) \\ y(w) \\ (1/2 \cosh(\alpha) \beta^2 + \sinh(\alpha) (1 + 1/2 \beta^2)) t(w) \\ (1/2 \sinh(\alpha) \beta^2 + \cosh(\alpha) (1 + 1/2 \beta^2)) t(w) \end{pmatrix}. \quad (7.97)$$

and the first fundamental form is given in (6.99) by:

$$I_{\Sigma^3} = \begin{pmatrix} y'^2(w) - t'^2(w) & 0 & 0 \\ 0 & t^2(w) & \beta t^2(w) \\ 0 & \beta t^2(w) & t^2(w)(1 + \beta^2) \end{pmatrix}. \quad (7.98)$$

Now, if we apply the condition of  $y'^2(w) - t'^2(w) = -1$ , then we have :

$$I_{\Sigma^3} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & t^2(w) & \beta t^2(w) \\ 0 & \beta t^2(w) & t^2(w)(1 + \beta^2) \end{pmatrix}. \quad (7.99)$$

So,

$$-d^2w + t^2(w)d^2\beta + t^2(w)\beta(d\beta)(d\alpha) + t^2(w)(1 + \beta^2)d^2\alpha. \quad (7.100)$$

From the first fundamental form, we have the Lagrangian given by:

$$-\dot{w}^2 + t^2(w)\dot{\beta}^2 + t^2(w)\beta\dot{\beta}\dot{\alpha} + t^2(w)(1 + \beta^2)\dot{\alpha}^2. \quad (7.101)$$



A geodesic on the surface is given by the Euler Lagrangian equation:

$$\frac{d}{ds} \left( \frac{\partial L}{\partial \dot{w}} \right) = \frac{\partial L}{\partial w}, \quad (7.102)$$

which gives:

$$\ddot{w} = -t(w)t'(w)\dot{\beta}^2 - t(w)t'(w)\beta\dot{\beta}\dot{\alpha} - t(w)t'(w)(1 + \beta^2)\dot{\alpha}^2. \quad (7.103)$$

Also,

$$\frac{d}{ds} \left( \frac{\partial L}{\partial \dot{\alpha}} \right) = \frac{\partial L}{\partial \alpha} = 0, \quad (7.104)$$

this means

$$\frac{\partial L}{\partial \dot{\alpha}} = t^2(w)\beta\dot{\beta} + 2t^2(w)(1 + \beta^2)\dot{\alpha} \quad (7.105)$$

is constant along the geodesic.

And,

$$\frac{d}{ds} \left( \frac{\partial L}{\partial \dot{\beta}} \right) = \frac{\partial L}{\partial \beta} = t^2(w)\dot{\beta}\dot{\alpha} + 2t^2(w)\dot{\beta}\dot{\alpha}^2 \neq 0. \quad (7.106)$$

So,

$$\frac{\partial L}{\partial \dot{\beta}} \quad (7.107)$$

is not constant.

Now, as previous section we change the coordinate of this equation into  $u, v$  terms, as defined in chapter 6 (6.4.3).

recall the relationship between  $u, v$  terms and  $\alpha, \beta$  terms is given by :

$$\alpha = v \quad , \quad \beta = ue^{-v}, \quad (7.108)$$

or

$$u = \beta e^{\alpha} \quad , \quad v = \alpha. \quad (7.109)$$

Thus the Lagrangian now is given by:

$$-\dot{w}^2 + t^2(w)e^{-2v}\dot{u}^2 + t^2(w)\dot{v}^2 \quad (7.110)$$

which has a constant along the geodesic given by :

$$\frac{d}{ds} \left( \frac{\partial L}{\partial \dot{u}} \right) = \frac{\partial L}{\partial u} = 0, \quad (7.111)$$

which means:

$$\frac{\partial L}{\partial \dot{u}} = 2t^2(w)e^{-2v}\dot{u} \quad (7.112)$$

is constant along the geodesic.

Again, we express this constant to  $\beta, \alpha$  terms, then

$$\frac{\partial L}{\partial \dot{u}} = 2t^2(w)e^{-\alpha} (\beta\dot{\alpha} + \dot{\beta}) \quad (7.113)$$

is constant.

It follows that the time-like geodesics are given by :

$$\begin{aligned} \ddot{w} &= -t(w)t'(w)\dot{\beta}^2 - t(w)t'(w)\beta\dot{\beta}\dot{\alpha} - t(w)t'(w)(1 + \beta^2)\dot{\alpha}^2, \\ \frac{d}{ds} \left( t^2(w)\beta\dot{\beta} + 2t^2(w)(1 + \beta^2)\dot{\alpha} \right) &= 0 \\ \frac{d}{ds} \left( 2t^2(w)e^{-\alpha} (\beta\dot{\alpha} + \dot{\beta}) \right) &= 0. \end{aligned} \quad (7.114)$$

Now, we have two conserved quantities on this parametrization, this time we are going to work out that these conserved quantities are not commute.

Now, this system has the conserved quantities given by:

$$P = t^2(w)\beta\dot{\beta} + 2t^2(w)(1 + \beta^2)\dot{\alpha} \quad \text{and} \quad Q = 2t^2(w)e^{-\alpha} (\beta\dot{\alpha} + \dot{\beta}). \quad (7.115)$$

So, by using the Poisson bracket which given in (2.34) by:

$$[f, g] = \sum_{\alpha=1}^n \left( \frac{\partial f}{\partial q_{\alpha}} \frac{\partial g}{\partial p_{\alpha}} - \frac{\partial f}{\partial p_{\alpha}} \frac{\partial g}{\partial q_{\alpha}} \right). \quad (7.116)$$

Then, at this time  $f = P = t^2(w)\beta\dot{\beta} + 2t^2(w)(1 + \beta^2)\dot{\alpha}$  and  $g = Q = 2t^2(w)e^{-\alpha}(\beta\dot{\alpha} + \dot{\beta})$  we obtain

$$\begin{aligned} [P, Q] &= \frac{\partial P}{\partial \beta} \frac{\partial Q}{\partial P} + \frac{\partial P}{\partial \alpha} \frac{\partial Q}{\partial Q} - \frac{\partial P}{\partial P} \frac{\partial Q}{\partial \beta} - \frac{\partial P}{\partial Q} \frac{\partial Q}{\partial \alpha} \\ &= (t^2(w)\dot{\beta} + 4t^2(w)\beta\dot{\alpha})(0) + (0)(1) - (1)(2t^2(w)e^{-\alpha}\dot{\alpha}) - (0)(1) \\ &= -2t^2(w)e^{-\alpha}\dot{\alpha} \neq 0. \end{aligned} \quad (7.117)$$

Thus, the conserved quantities are not commute, as a result the system is not integrable. So the geodesics are not in terms of integrals.

In conclusion, we still have the conserved quantities, but in this case they depend explicitly on  $\beta$  and  $\alpha$  as well as on  $w$ .

### 7.3 Surface of Rotation Generated by Spherical Symmetric Case

In this section we give a brief discussion of the surface of rotation which generated by spherical symmetric. We then will have the conserved quantities of this surface of rotation, but again the conserved quantities do not commute.

Recall the surface of rotation of this case which given in chapter 6 equation (6.108) by

$$\Sigma^4(w, u, v) = \begin{pmatrix} \cos(u) \sin(v) z(w) \\ \cos(u) \cos(v) z(w) \\ \sin(u) z(w) \\ t(w) \end{pmatrix}. \quad (7.118)$$

and the first fundamental form of this surface is given in (6.113) by:

$$I_{\Sigma^4} = \begin{pmatrix} z'^2(w) - t'^2(w) & 0 & 0 \\ 0 & z^2(w) & 0 \\ 0 & 0 & \cos^2(u)z^2(w) \end{pmatrix}. \quad (7.119)$$

And if we apply the condition of  $z'^2(w) - t'^2(w) = -1$ , then we have:

$$I_{\Sigma^4} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & z^2(w) & 0 \\ 0 & 0 & \cos^2(u)z^2(w) \end{pmatrix}. \quad (7.120)$$

From the first fundamental form, we have the Lagrangian given by:

$$L = -\dot{w}^2 + z^2(w)\dot{u}^2 + z^2(w)\cos^2(u)\dot{v}^2. \quad (7.121)$$

A geodesic on the surface is given by the Euler Lagrangian Equation:

$$\frac{d}{ds} \left( \frac{\partial L}{\partial \dot{w}} \right) = \frac{\partial L}{\partial w}, \quad (7.122)$$

which gives :

$$\ddot{w} = -z(w)z'(w)\dot{u}^2 - z(w)z'(w)\cos^2(u)\dot{v}^2. \quad (7.123)$$

Also,

$$\frac{d}{ds} \left( \frac{\partial L}{\partial \dot{v}} \right) = \frac{\partial L}{\partial v} = 0, \quad (7.124)$$

which means:

$$\frac{\partial L}{\partial \dot{v}} = 2z^2(w)\cos^2(u)\dot{v} \quad (7.125)$$

is constant along the geodesic

And,

$$\frac{d}{ds} \left( \frac{\partial L}{\partial \dot{u}} \right) = \frac{\partial L}{\partial u} = -2z^2(w)\cos(u)\sin(u)\dot{v}^2 \neq 0, \quad (7.126)$$

So,

$$\frac{\partial L}{\partial \dot{v}} \quad (7.127)$$

is not constant

Now, as in the previous section, we are seeking the second conserved quantity from the other parametrization of this surface which is given in (6.116) by

$$\Sigma^{4x} \begin{pmatrix} \sin(\alpha) z(w) \\ \cos(\alpha) \cos(\beta) z(w) \\ \cos(\alpha) \sin(\beta) z(w) \\ t(w) \end{pmatrix}. \quad (7.128)$$

With the first fundamental form given in (6.117) by:

$$I_{\Sigma^{4x}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & z^2(w) & 0 \\ 0 & 0 & \cos^2(\alpha) z^2(w) \end{pmatrix}. \quad (7.129)$$

Also same as above, we can write straightforward the Lagrangian :

$$-\dot{w}^2 + z^2(w)\dot{\alpha}^2 + z^2(w)\cos^2(\alpha)\dot{\beta}^2, \quad (7.130)$$

which has a conserved quantity along the geodesic given:

$$\frac{d}{ds} \left( \frac{\partial L}{\partial \dot{\beta}} \right) = \frac{\partial L}{\partial \beta} = 0, \quad (7.131)$$

which means:

$$\frac{\partial L}{\partial \dot{\beta}} = 2z^2(w)\cos^2(\alpha)\dot{\beta} \quad (7.132)$$

is constant along the geodesic

Now we recall the relationship between the two parametrization given in (6.118) and

(6.119) by:

$$u = \arcsin(\cos(\alpha) \sin(\beta)) \quad \text{and} \quad v = \arcsin\left(\frac{\sin(\alpha)}{\sqrt{1 - (\cos(\alpha))^2 (\sin(\beta))^2}}\right) \quad (7.133)$$

or

$$\alpha = \arcsin(\cos(u) \sin(v)) \quad \text{and} \quad \beta = \arcsin\left(\frac{\sin(u)}{\sqrt{1 - (\cos(u))^2 (\sin(v))^2}}\right). \quad (7.134)$$

Therefore, the constant

$$\frac{\partial L}{\partial \dot{\beta}} = 2z^2(w) \cos^2(\alpha) \dot{\beta} \quad (7.135)$$

can be given by using this relation and using Maple :

$$2z^2(w) \cos^2[\arcsin(\cos(u) \sin(v))] \frac{d}{ds} \left[ \arcsin\left(\frac{\sin(u)}{\sqrt{1 - (\cos(u))^2 (\sin(v))^2}}\right) \right]. \quad (7.136)$$

This is the second conserved quantity of this surface.

It follows that the time-like geodesics are given by :

$$\begin{aligned} \ddot{w} &= -z(w)z'(w)\dot{u}^2 - z(w)z'(w)\cos^2(u)\dot{v}^2, \\ \frac{d}{ds} (2z^2(w)\cos^2(u)\dot{v}) &= 0 \\ \frac{d}{ds} \left( 2z^2(w)\cos^2[\arcsin(\cos(u)\sin(v))] \frac{d}{ds} \left[ \arcsin\left(\frac{\sin(u)}{\sqrt{1 - (\cos(u))^2 (\sin(v))^2}}\right) \right] \right) &= 0. \end{aligned} \quad (7.137)$$

Finally, we have two conserved quantities on this parametrization. However, these two conserved quantities do not commute, since the generators of rotations of the sphere about different axes do not commute. In other words, it is same situation as previous section of the surface parametrized by  $\Sigma^2$  and  $\Sigma^3$ .

In conclusion, we still have the conserved quantities, but they do not commute, as a

result the system is not integrable.

#### 7.4 Clairaut's Theorem of Surface of Rotation Generated by Boost and Rotation Subgroups

As before we consider the case of the time-like generator, as that of the space-like generator gives the same result.

Recall the Surface of rotation parametrized by rotation and boost given in chapter six equation (6.124). This surface has no axis of rotation, but is closely related to the boosts and spatial rotations in  $\mathbb{M}^{2,1}$ .

It is parametrized by:

$$\Sigma^5(w, u, v) = \mathcal{M}_3(u) \cdot \mathcal{M}_4(v) \cdot \gamma(w) = \begin{pmatrix} -\sin(v)y(w) \\ \cos(v)y(w) \\ \sinh(u)t(w) \\ \cosh(u)t(w) \end{pmatrix}. \quad (7.138)$$

This time, the first fundamental form is given by :

$$I_{\Sigma^5} = \begin{pmatrix} y'^2(w) - t'^2(w) & 0 & 0 \\ 0 & t^2(w) & 0 \\ 0 & 0 & y^2(w) \end{pmatrix}. \quad (7.139)$$

Now, if we apply the condition of  $y'^2(w) - t'^2(w) = -1$ , then we have:

$$I_{\Sigma^5} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & t^2(w) & 0 \\ 0 & 0 & y^2(w) \end{pmatrix}. \quad (7.140)$$

So,

$$-d^2w + t^2(w)d^2u + y^2(w)d^2v. \quad (7.141)$$

So , we have the Lagrangian :

$$- \dot{w}^2 + t^2(w)\dot{u}^2 + y^2(w)\dot{v}^2, \quad (7.142)$$

given an Euler-Lagrange equations

$$\begin{aligned} \ddot{w} &= -t(w)t'(w)\dot{u}^2 - y(w)y'(w)\dot{v}^2, \\ \frac{d}{ds}(t^2(w)\dot{u}) &= 0, \\ \frac{d}{ds}(y^2(w)\dot{v}) &= 0 \end{aligned} \quad (7.143)$$

Now we consider the curve  $\gamma(s)$  a time-like curve geodesic on the surface of revolution, so it is given by:  $w(s), u(s), v(s)$ .

Then, we can see that

$$\dot{\gamma}(s) = \dot{w}\Sigma_w^5 + \dot{u}\Sigma_u^5 + \dot{v}\Sigma_v^5. \quad (7.144)$$

Again we can notice that,  $\Sigma_w^5 = n_w$  is a unit time-like vector pointing along the meridians, while  $\Sigma_u^5 = t(w)n_u$  where  $n_u$  is a unit space-like vector pointing along  $u$ - axis of the parallels, and  $\Sigma_v^5 = y(w)n_v$  where  $n_v$  is a unit space-like vector pointing along  $v$ - axis of the parallels. It follows that the plane spanned by  $n_u$  and  $n_v$  is space-like, with  $n_u$  and  $n_v$  providing an orthonormal basis. Furthermore,  $g(\Sigma_w^5, \Sigma_u^5) = g(\Sigma_w^5, \Sigma_v^5) = g(\Sigma_u^5, \Sigma_v^5) = 0$ ; so we have an orthonormal basis.

Therefore,

$$\dot{\gamma}(s) = \dot{w}n_w + \dot{u}t(w)n_u + \dot{v}y(w)n_v. \quad (7.145)$$

We define a space-like vector  $n_{w^\perp}$  given by :

$$n_{w^\perp} = \frac{\dot{u}t(w)n_u + \dot{v}y(w)n_v}{\|\dot{u}t(w)n_u + \dot{v}y(w)n_v\|} \quad (7.146)$$



which is a unit vector perpendicular to  $n_w$ .

So, we have:

$$\dot{\gamma}(s) = n_w \cosh(\theta) + n_{w^\perp} \sinh(\theta), \quad (7.147)$$

such that,  $n_{w^\perp}$  is a unit vector pointing the perpendicular projection of  $n_w$  of the surface of rotation, which has  $n_w$  as a unit vector along the meridians.

If  $\phi$  is the angle between  $n_u$  and  $n_{w^\perp}$  then:

$$\dot{\gamma}(s) = n_w \cosh(\theta) + [n_u \cos(\phi) + n_v \sin(\phi)] \sinh(\theta). \quad (7.148)$$

From equations (7.145) and (7.148), we have :

$$t(w)\dot{u} = \cos(\phi) \sinh(\theta) \quad \text{and} \quad y(w)\dot{v} = \sin(\phi) \sinh(\theta) \quad (7.149)$$

which

$$t^2(w)\dot{u} = t(w) \cos(\phi) \sinh(\theta) \quad \text{and} \quad y(w)^2\dot{v} = y(w) \sin(\phi) \sinh(\theta) \quad (7.150)$$

We can conclude that; the second and third Euler-Lagrangian equation is equivalent to  $t(w) \cos \phi \sinh(\theta)$  and  $y(w) \sin \phi \sinh(\theta)$  being constants.

Also as before, if we constrain one parameter in (7.150) say  $v = k$  where  $k$  is constant. We get  $\phi = 0$ , then  $t(w) \sinh(\theta)$  is constant. Also if we restrict  $u = k$ . Then  $\phi = \frac{\pi}{2}$ , so  $y(w) \sinh(\theta)$  is again constant. This discussion recovers the case of the rotation and boost in  $\mathbb{M}^{2,1}$ .

Conversely, let  $\gamma$  be a proper time parametrization curve, such that  $t(w) \cos \phi \sinh(\theta) = t^2(w)\dot{u}$  and  $y(w) \sin \phi \sinh(\theta) = y^2(w)\dot{v}$  are constants, and  $\dot{w} \neq 0$ .

So,

$$\dot{w}^2 - (t^2(w)\dot{u}^2 + y^2(w)\dot{v}^2) = 1 \quad \text{and} \quad t^2(w)\dot{u}, y^2(w)\dot{v} \text{ are constants}$$

Differentiating this with respect to  $s$ , we have :

$$2\dot{w}\ddot{w} - 2tt'\dot{w}\dot{u}^2 - 2t^2\dot{w}\ddot{u} - 2yy'\dot{w}\dot{v}^2 - 2y^2\dot{w}\ddot{v} = 0, \quad (7.151)$$

or

$$\dot{w}\ddot{w} - tt'\dot{w}\dot{u}^2 - yy'\dot{w}\dot{v}^2 - t^2\dot{w}\ddot{u} - y^2\dot{w}\ddot{v} = 0 \quad (7.152)$$

Now, from:

$$\frac{d}{ds}(t^2(w)\dot{u}) = 0 = 2tt'\dot{w}\dot{u} + t^2\ddot{u} \quad (7.153a)$$

$$\frac{d}{ds}(y^2(w)\dot{v}) = 0 = 2yy'\dot{w}\dot{v} + y^2\ddot{v} \quad (7.153b)$$

Now, multiplying (7.153a) by  $\dot{u}$ , and (7.153b) by  $\dot{v}$ , then add them together, we obtain:

$$2tt'\dot{w}\dot{u}^2 + 2yy'\dot{w}\dot{v}^2 + t^2\dot{w}\ddot{u} + y^2\dot{w}\ddot{v} = 0. \quad (7.154)$$

Adding (7.152) to (7.154) gives:

$$\dot{w}\ddot{w} + t(w)t'(w)\dot{w}\dot{u}^2 + y(w)y'(w)\dot{w}\dot{v}^2 = 0, \quad (7.155)$$

or

$$\dot{w}\ddot{w} = -t(w)t'(w)\dot{w}\dot{u}^2 - y(w)y'(w)\dot{w}\dot{v}^2. \quad (7.156)$$

And,  $\dot{w} \neq 0$ , so

$$\ddot{w} = -t(w)t'(w)\dot{u}^2 - y(w)y'(w)\dot{v}^2; \quad (7.157)$$

which is the first Euler-Lagrangian equation. It follows that  $\gamma(s)$  is time-like geodesic.

□

Now, one can define the Hamiltonian version of the Lagrangian equation. So recalling

equations (7.142) we have Lagrangian equation given by:

$$L = \frac{1}{2} (-\dot{w}^2 + t^2(w)\dot{u}^2 + y^2(w)\dot{v}^2) \quad (7.158)$$

and the partial derivatives of all components of  $L$  are

$$\frac{\partial L}{\partial \dot{w}} = P_w = -\dot{w} \quad , \quad \frac{\partial L}{\partial \dot{u}} = P_u = t^2(w)\dot{u} \quad \text{and} \quad \frac{\partial L}{\partial \dot{v}} = P_v = y^2(w)\dot{v} \quad (7.159)$$

Also from equation (7.143) we have  $P_u$  and  $P_v$  are constants.

And,

$$\dot{w} = -P_w \quad , \quad \dot{u} = P_u/t^2 \quad \text{and} \quad \dot{v} = P_v/y^2 \quad (7.160)$$

Now, the Hamiltonian function for this case it is :

$$H = P_w \dot{w} + P_u \dot{u} + P_v \dot{v} - \frac{1}{2} (-\dot{w}^2 + t^2(w)\dot{u}^2 + y^2(w)\dot{v}^2) \quad (7.161)$$

substituting using (7.160) we obtain

$$H = \frac{1}{2} \left( -P_w + \frac{P_u^2}{t^2(w)} + \frac{P_v^2}{y^2(w)} \right). \quad (7.162)$$

As before, we just need to proof that the two conserved quantities of the Hamiltonian function  $P_u$  and  $P_v$  are commute.

So, by using the Poisson bracket (2.34) we obtain

$$\begin{aligned} [P_u, P_v] &= \frac{\partial P_u}{\partial u} \frac{\partial P_v}{\partial P_u} + \frac{\partial P_u}{\partial v} \frac{\partial P_v}{\partial P_v} - \frac{\partial P_u}{\partial P_u} \frac{\partial P_v}{\partial u} - \frac{\partial P_u}{\partial P_v} \frac{\partial P_v}{\partial v} \\ &= (0)(0) + (0)(1) - (1)(0) - (0)(0) = 0. \end{aligned} \quad (7.163)$$

Thus, the conserved quantities commute, then the system is integrable. So the geodesics can be expressed in terms of integrals and solutions of algebraic equations.

Hence, as in section (7.1) we have the Lagrangian:

$$L = \frac{1}{2} (-\dot{w}^2 + t^2(w)\dot{u}^2 + y^2(w)\dot{v}^2) \quad (7.164)$$

with two constants:

$$\frac{d}{ds}(t^2(w)\dot{u}) = 0 \quad \text{and} \quad \frac{d}{ds}(y^2(w)\dot{v}) = 0 \quad (7.165)$$

then,

$$t^2(w)\dot{u} = \Omega_1 \quad \text{and} \quad y^2(w)\dot{v} = \Omega_2, \quad (7.166)$$

where  $\Omega_1$  and  $\Omega_2$  are constants.

So,

$$\dot{u} = \Omega_1/t^2(w) \quad \text{and} \quad \dot{v} = \Omega_2/y^2(w) \quad (7.167)$$

and so,

$$L = \frac{1}{2} \left( -\dot{w}^2 + \frac{\Omega_1^2}{t^2(w)} + \frac{\Omega_2^2}{y^2(w)} \right) \quad (7.168)$$

giving

$$\dot{w}^2 = \frac{\Omega_1^2}{t^2(w)} + \frac{\Omega_2^2}{y^2(w)} - 2L \quad (7.169)$$

or

$$\dot{w} = \sqrt{\frac{\Omega_1^2}{t^2(w)} + \frac{\Omega_2^2}{y^2(w)} - 2L} \quad (7.170)$$

rearranging this

$$\frac{dw}{\sqrt{\frac{\Omega_1^2}{t^2(w)} + \frac{\Omega_2^2}{y^2(w)} - 2L}} = ds \quad (7.171)$$

so that

$$s = \int \frac{dw}{\sqrt{\frac{\Omega_1^2}{t^2(w)} + \frac{\Omega_2^2}{y^2(w)} - 2L}} + C_1 \quad (7.172)$$

This specifies  $w$  as a function of  $s$

we now return to

$$\dot{u} = \Omega_1/t^2(w) \quad \text{and} \quad \dot{v} = \Omega_2/y^2(w) \quad (7.173)$$

and  $w$  is a function of  $s$  obtained above. Then

$$\dot{u} = \Omega_1/t^2(w(s)) \quad \text{and} \quad \dot{v} = \Omega_2/y^2(w(s)) \quad (7.174)$$

so that

$$u = \int \frac{\Omega_1}{t^2(w(s))} ds + C_2 \quad \text{and} \quad v = \int \frac{\Omega_2}{y^2(w(s))} ds + C_3 \quad (7.175)$$

which give all  $w, u$  and  $v$  explicitly in terms of integrals.

### 7.5 Conclusion

To summarise, we can see that, Clairaut's theorem has 4D Minkowski space analogues, but not a single analogue. In fact, the surface  $\Sigma^1$  has analogue of Clairaut's theorem with two conserved quantities along a time-like geodesic of the form  $\rho(w) \sinh(\theta)$ , where  $\rho(w) > 0$  is combining  $z(w)$  and  $t(w)$ . The surface  $\Sigma^2$  has a weaker analogue of Clairaut's theorem with two conserved quantities along a time-like geodesic but which are not obvious analogues of angular momentum. Finally the surface  $\Sigma^5$  which does not fix any axis, has two conserved quantities of  $t(w) \sinh(\theta)$  in direction of  $u$  and  $y(w) \sinh(\theta)$  in direction of  $v$ . In each case the conserved quantities determine the time-like geodesic, and in the first and third cases they have a similar interpretation to the Euclidean case, but not so obviously in the second.



## 8. CONCLUSION AND THE FUTURE WORK

To sum up, this thesis generalizes Clairaut's theorem to three and four dimensional Minkowski spaces. It begins first by introducing classical differential geometry of three dimensional Euclidean space, and reviews Clairaut's theorem of surfaces of revolution which define a well-known characterization of geodesics on a surface of revolution.

In Minkowski spaces however, which is the setting for this work, we distinguish three types of vectors space-like, time-like and null. Therefore, there are three distinct types of axes of rotations; space-like, time-like and null. More explicitly, there are three types of one parameter subgroup of isometries of Minkowski Spaces each of which leaves a line (axis) pointwise fixed. We consider the rigid motion of the ambient space that makes the straight line fixed. So we investigate the corresponding rotation for each case. Therefore, we generate matrices of rotations corresponding to each axis of rotation. Thus, in three dimensional Minkowski space, we classified three types of rotations. Also the classification and characterization of the rotational surfaces of three dimension Minkowski space.

Next we generalized Clairaut's theorem to these surfaces of rotation, in this case of time-like geodesics. We see that Clairaut's theorem has a three dimensional Minkowski space analogue with  $\rho \sinh \psi$  replacing  $\rho \sin \psi$  as the quantity conserved along a time-like geodesic. In addition, we see that for small values of  $\psi$ , the geodesics will be close to those for the Euclidean case.

Interestingly, Clairaut's theorem in the different surfaces of rotations of Minkowski space seems to be the same. Although the meaning of distance from the axis of rotation varies, we have the same formal statement on each case, which can be thought of as a conservation of angular momentum.

Also in comparison to the Euclidean case, the characterisation of geodesics in surfaces of revolution looks formally identical in the Euclidean and Minkowskian cases: in each case geodesics are completely characterized by  $\rho^2\dot{v}$  being a conserved quantity. In spite of this, the difference in signature results in entirely different qualitative behaviour of the geodesics in these surfaces.

Building on this, in four dimensional Minkowski space we seek a two parameter subgroup of special orthogonal matrices of four dimensions  $SO(3,1)$  which are analogues of rotations in  $\mathbb{E}^3$ . So, we found two parameter subgroups which fix some axes of rotation. Then we found also three different cases of two dimensional sub-algebras. Hence, corresponding to these types of two dimensional subgroups of isometries we generate special cases of surfaces of rotations of four dimensional Minkowski space.

Finally, we consider the generalization Clairaut's theorem to these cases. It can be seen that, Clairaut's theorem has a four dimensional Minkowski space analogue. For more explanation, in the first surface of rotation, the geodesics are completely characterized by  $\rho^2(w)\dot{u}$  and  $\rho^2(w)\dot{v}$  being two conserved quantities. And the second surface of rotation, the geodesics are completely characterized by  $t^2(w)e^{-2v}\dot{u}$  and  $t^2(w)\dot{v}$  being two conserved quantities, but with a less clear interpretation as analogues of angular momentum. In the third surface of rotation, the geodesics are completely characterized by  $t^2(w)\dot{u}$  and  $y^2(w)\dot{v}$  being two conserved quantities, in this case because this surface of rotation does not fix any axis of rotation, just the origin as a point.

In brief, these conserved quantities have a similar interpretation to the Euclidean case, and determine the geodesics.

This work suggests several avenues of future investigation. Throughout this thesis for both three and four dimensional Minkowski spaces we have used a curve to generate Lorentzian surfaces of rotations, and consider time-like geodesics.

In the future work, we will distinguish surfaces of rotations for other generators and



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consider space-like and null surfaces, then try to generalize Clairaut's theorem including the possibility of the geodesics being space-like or null. Moreover, the Jacobi Field: is a vector field along a geodesic  $\gamma$  describing the difference between the geodesic and an "infinitesimally close" geodesic. We will try to think about geodesic deviation in surfaces of rotation in Minkowski space.



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